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Three-Body 3D-Kepler Electromagnetic Problem—Existence of Periodic Solutions

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Abstract: The main purpose of the present paper is to prove the existence of periodic solutions of the three-body problem in the 3D Kepler formulation. We have solved the same problem in the case when the three particles are considered in an external inertial system. We start with the three-body equations of motion, which are a subset of the equations of motion (previously derived by us) for any number of bodies. In the Minkowski space, there are 12 equations of motion. It is proved that three of them are consequences of the other nine, so their number becomes nine, as much as the unknown trajectories are. The Kepler formulation assumes that one particle (the nucleus) is placed at the coordinate origin. The motion of the other two particles is described by a neutral system with respect to the unknown velocities. The state-dependent delays arise as a consequence of the finite vacuum speed of light. We obtain the equations of motion in spherical coordinates and split them into two groups. In the first group all arguments of the unknown functions are delays. We take their solutions as initial functions. Then, the equations of motion for the remaining two particles must be solved to the right of the initial point. To prove the existence–uniqueness of a periodic solution, we choose a space consisting of periodic infinitely smooth functions satisfying some supplementary conditions. Then, we use a suitable operator which acts on these spaces and whose fixed points are periodic solutions. We apply the fixed point theorem for the operators acting on the spaces of periodic functions. In this manner, we show the stability of the He atom in the frame of classical electrodynamics. In a previous paper of ours, we proved the existence of spin functions for plane motion. Thus, we confirm the Bohr and Sommerfeld's hypothesis for the He atom.

Keywords: classical electrodynamics; three-body problem; periodic solutions; fixed point method; neutral system

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1. Introduction

The primary goal of the present paper is to investigate the three-body problem of classical electrodynamics in the 3D Kepler formulation. In [1], we formulated the same three-body problem externally for the particle inertial system, and in [2], we proved the existence–uniqueness of a periodic solution. The three-body problem is a particular case of the general formulation of the N-body problem (cf. [3]), where the existence–uniqueness of escape trajectories of N charged particles has been proved. Here, we add radiation terms for each of the three particles. We use the relativistically invariant form of the Dirac radiation term introduced in [4]. We proceed from the system from [1] and passing to the spherical coordinates, we formulate the 3D Kepler problem for three charged particles. The results obtained allow us to extend qualitative investigations concerning He atom and He-like atoms such as Li^{+}, and others. The method for solving the equations of motion is based on the choice of suitable spaces of functions consisting of an infinitely smooth periodic satisfying some additional conditions and introducing operators acting on these spaces. Their fixed points are periodic solutions of the equations of motion. It is not superfluous to note that, using the 3D Kepler formulation, we put the nuclei at the origin and look for trajectories of two moving particles circling around the nuclei. From a mathematical point of view, it is a formal operation, but from a physical point of view, this means that we pass from the relativistic case to the quasi-classical one, because the Kepler problem is in fact a classical (Newton) one. This is also confirmed by the type of radiation term, which turns out to be in the Lorentz form.

This paper consists of seven sections, five appendices and conclusions. Section 1 is the introduction. Section 2 contains the 3D Kepler formulation for the equations of motion in Cartesian coordinates. This leads to the separation of the system into two groups. One of them consists of three equations containing unknown functions on the initial interval. We call them initial equations. We follow R.D. Driver's approach (cf. [5]) to describe the retarded interaction between particles, which is based on the idea of W. Pauli [6] and J.L. Synge [7]. The state-dependent delays are obtained. In fact, with respect to the unknown velocities, the system obtained is a neutral type, and the delays depend on the unknown trajectories. In Section 3, we derive the basic equations of motion in spherical coordinates. The equations obtained are again of a neutral type, with delays depending on the unknown trajectories. In Section 4, we introduce the function spaces consisting of infinitely differentiable periodic functions which satisfy some additional conditions. Some auxiliary propositions are also proved. In Section 5, we introduce operators whose fixed points are periodic solutions to the equations of motion and prove the Main Lemma. Section 6 contains the main result, which guarantees the existence–uniqueness of the periodic solution to equations of motion for the three-body problem in the 3D Kepler form. The proof is based on a fixed point theorem [8]. A numerical example is given. In Section 7, the Bohr and Sommerfeld hypotheses are discussed.

We mention some results in which different approaches are applied [9–18].

We recall some denotations and results from [1,2] concerning the three-body problem in Cartesian coordinates. We consider the system of 12 equations of motion using radiation terms for the three-body system [1]:

$$\begin{aligned} m_1 \frac{d\lambda_r^{(1)}}{ds_1} &= \frac{e_1}{c^2}\left(F_{rl}^{(12)}\lambda_l^{(1)} + F_{rl}^{(13)}\lambda_l^{(1)} + F_{rl}^{(1)rad}\lambda_l^{(1)}\right), \\ m_2 \frac{d\lambda_r^{(2)}}{ds_2} &= \frac{e_2}{c^2}\left(F_{rl}^{(21)}\lambda_l^{(2)} + F_{rl}^{(23)}\lambda_l^{(2)} + F_{rl}^{(2)rad}\lambda_l^{(2)}\right), \\ m_3 \frac{d\lambda_r^{(3)}}{ds_3} &= \frac{e_3}{c^2}\left(F_{rl}^{(31)}\lambda_l^{(3)} + F_{rl}^{(32)}\lambda_l^{(3)} + F_{rl}^{(3)rad}\lambda_l^{(3)}\right), \end{aligned} \tag{1}$$

where $r = 1,2,3,4$; c is the vacuum speed of light; m_k are the masses; and $e_k (k = 1,2,3)$ are the charges of the particles. Recall that there is a summation in repeating l in the right-hand sides of the above system. The elements of the electromagnetic tensors $F_{rl}^{(kn)} = \frac{\partial \Phi_l^{(n)}}{\partial x_r^{(k)}} - \frac{\partial \Phi_r^{(n)}}{\partial x_l^{(k)}}$ can be calculated by the retarded Lienard–Wiechert potentials $\Phi_r^{(n)} = -\frac{e_n \lambda_r^{(n)}}{\langle \lambda^{(n)}, \xi^{(kn)} \rangle_4}$, while the radiation terms can be calculated as a half of the difference between the retarded and advance potentials, in accordance with the Dirac assumption:

$$F_{mn}^{(k)rad} = \frac{1}{2}\left[\left(\frac{\partial A_n^{(k)ret}}{\partial x_m^{(k)ret}} - \frac{\partial A_m^{(n)ret}}{\partial x_n^{(k)ret}}\right) - \left(\frac{\partial A_n^{(k)adv}}{\partial x_m^{(k)adv}} - \frac{\partial A_m^{(n)adv}}{\partial x_n^{(k)adv}}\right)\right],$$

where $A_n^{(k)ret} = -\frac{e_k \lambda_n^{(k)ret}}{\langle \lambda^{(k)ret}, \xi^{(k)ret} \rangle_4}$, $A_n^{(k)adv} = -\frac{e_k \lambda_n^{(k)adv}}{\langle \lambda^{(k)adv}, \xi^{(k)adv} \rangle_4}$.

Here, $\left(x_1^{(k)}(t), x_2^{(k)}(t), x_3^{(k)}(t), x_4^{(k)}(t) = ict\right)$, $(k = 1,2,3)$ are space–time coordinates of the charged particles. The dot product in the Minkowski space is $\langle a, b \rangle_4 = a_n b_n = \sum_{n=1}^{4} a_n b_n$, while in the three-dimensional subspace, $\langle a,b\rangle = a_\alpha b_\alpha = \sum_{\alpha=1}^{3} a_\alpha b_\alpha$. The elements of proper times are ds_k $(k=1,2,3)$ and the unit tangent vectors to the world lines are

$$\lambda^{(k)} = \left(\lambda_1^{(k)},\lambda_2^{(k)},\lambda_3^{(k)},\lambda_4^{(k)}\right) = \left(\frac{u_1^{(k)}(t)}{\Delta_k},\frac{u_2^{(k)}(t)}{\Delta_k},\frac{u_3^{(k)}(t)}{\Delta_k},\frac{ic}{\Delta_k}\right),\ \vec{u}^{(k)} = \left(u_1^{(k)}(t),u_2^{(k)}(t),u_3^{(k)}(t)\right),$$
$$\Delta_k = \sqrt{c^2-\left\langle \vec{u}^{(k)},\vec{u}^{(k)}\right\rangle}.$$

We have proved that three equations from (1) are a consequence of the rest [1].

The equations of motion become nine in number, as do the unknown functions $\left(u_1{}^{(k)}(t),u_2{}^{(k)}(t),\ u_3{}^{(k)}(t)\right),\ (k=1,2,3)$.

The isotropic vectors $\xi^{(km)}(k=1,2,3\,;\ m\neq k)$ are obtained in the following way: fix any event on the world line of the k-th particle and draw the light cone into the past. This cone intersects the world line of the m-th particle in any other (past) event. Then,

$$\xi^{(km)} = \left(\xi_1^{(km)},\xi_2^{(km)},\xi_3^{(km)},\xi_4^{(km)}\right) = \left(\vec{\xi}^{(km)},\xi_4^{(km)}\right) =$$
$$= \left(x_1^{(k)}(t)-x_1^{(m)}(t-\tau_{km}(t)),x_2^{(k)}(t)-x_2^{(m)}(t-\tau_{km}(t)),x_3^{(k)}(t)-x_3^{(m)}(t-\tau_{km}(t)),ic\tau_{km}(t)\right).$$

$\xi^{(km)}$ are isotropic four-vectors $\left\langle \xi^{(km)},\xi^{(km)}\right\rangle_4 = 0$, which implies that the retarded functions $\tau_{km}(t)$ should satisfy the following functional equations:

$$\tau_{km}(t) = \frac{1}{c}\sqrt{\left\langle \vec{\xi}^{(km)},\vec{\xi}^{(km)}\right\rangle} \equiv \frac{1}{c}\sqrt{\sum_{\alpha=1}^{3}\left[x_\alpha^{(k)}(t)-x_\alpha^{(m)}(t-\tau_{km}(t))\right]^2}.$$

The above equations are six in number because $(km)=(12),(13),(23),(21),(31),(32)$ and $u^{(m)} \equiv u^{(m)}(t-\tau_{km}(t))$, for $m\neq k$, $\Delta_{km} = \sqrt{c^2-\left\langle \vec{u}^{(m)}(t-\tau_{km}),\vec{u}^{(m)}(t-\tau_{km})\right\rangle}$,

$$\lambda^{(m)} = \left(\lambda_1^{(m)},\lambda_2^{(m)},\lambda_3^{(m)},\lambda_4^{(m)}\right) = \left(\frac{u_1^{(m)}(t-\tau_{km})}{\Delta_{km}},\frac{u_2^{(m)}(t-\tau_{km})}{\Delta_{km}},\frac{u_3^{(m)}(t-\tau_{km})}{\Delta_{km}},\frac{ic}{\Delta_{km}}\right) = \left(\frac{\vec{u}^{(m)}(t-\tau_{km})}{\Delta_{km}},\frac{ic}{\Delta_{km}}\right).$$

2. Separation of the Equations of Motion and Derivation of the Initial Equations

The assumption (C):$\left\|\vec{u}^{(k)}\right\| = \sqrt{\left\langle \vec{u}^{(k)}(t),\vec{u}^{(k)}(t)\right\rangle} \leq \bar{c} < c$ $(k=1,2,3;\ \bar{c}=const)$ implies $c^2-\left\langle \vec{u}^{(k)},\vec{u}^{(k)}\right\rangle \geq c^2-\bar{c}^2 > 0$. This condition allows us to obtain from (1) a system containing three groups of equations (cf. [1]):

$$\begin{aligned}
\dot{u}_1^{(1)} &= \frac{c^2-\left(u_1^{(1)}\right)^2}{c^2}\left(G_1^{(12)}+G_1^{(13)}+G_1^{(1)rad}\right)-\frac{u_1^{(1)}u_2^{(1)}}{c^2}\left(G_2^{(12)}+G_2^{(13)}+G_2^{(1)rad}\right)-\frac{u_1^{(1)}u_3^{(1)}}{c^2}\left(G_3^{(12)}+G_3^{(13)}+G_3^{(1)rad}\right)\equiv U_1^1\\
\dot{u}_2^{(1)} &= -\frac{u_1^{(1)}u_2^{(1)}}{c^2}\left(G_1^{(12)}+G_1^{(13)}+G_1^{(1)rad}\right)+\frac{c^2-\left(u_2^{(1)}\right)^2}{c^2}\left(G_2^{(12)}+G_2^{(13)}+G_2^{(1)rad}\right)-\frac{u_2^{(1)}u_3^{(1)}}{c^2}\left(G_3^{(12)}+G_3^{(13)}+G_3^{(1)rad}\right)\equiv U_2^1\\
\dot{u}_3^{(1)} &= -\frac{u_1^{(1)}u_3^{(1)}}{c^2}\left(G_1^{(12)}+G_1^{(13)}+G_1^{(1)rad}\right)-\frac{u_2^{(1)}u_3^{(1)}}{c^2}\left(G_2^{(12)}+G_2^{(13)}+G_2^{(1)rad}\right)+\frac{c^2-\left(u_3^{(1)}\right)^2}{c^2}\left(G_3^{(12)}+G_3^{(13)}+G_3^{(1)rad}\right)\equiv U_3^1,
\end{aligned} \tag{2}$$

$$\begin{aligned}
\dot{u}_1^{(2)} &= \frac{c^2-\left(u_1^{(2)}\right)^2}{c^2}\left(G_1^{(21)}+G_1^{(23)}+G_1^{(2)rad}\right)-\frac{u_1^{(2)}u_2^{(2)}}{c^2}\left(G_2^{(21)}+G_2^{(23)}+G_2^{(2)rad}\right)-\frac{u_1^{(2)}u_3^{(2)}}{c^2}\left(G_3^{(21)}+G_3^{(23)}+G_3^{(2)rad}\right)\equiv U_1^2\\
\dot{u}_2^{(2)} &= -\frac{u_1^{(2)}u_2^{(2)}}{c^2}\left(G_1^{(21)}+G_1^{(23)}+G_1^{(2)rad}\right)+\frac{c^2-\left(u_2^{(2)}\right)^2}{c^2}\left(G_2^{(21)}+G_2^{(23)}+G_2^{(2)rad}\right)-\frac{u_2^{(2)}u_3^{(2)}}{c^2}\left(G_3^{(21)}+G_3^{(23)}+G_3^{(2)rad}\right)\equiv U_2^2\\
\dot{u}_3^{(2)} &= -\frac{u_1^{(2)}u_3^{(2)}}{c^2}\left(G_1^{(21)}+G_1^{(23)}+G_1^{(2)rad}\right)-\frac{u_2^{(2)}u_3^{(2)}}{c^2}\left(G_2^{(21)}+G_2^{(23)}+G_2^{(2)rad}\right)+\frac{c^2-\left(u_3^{(2)}\right)^2}{c^2}\left(G_3^{(21)}+G_3^{(23)}+G_3^{(2)rad}\right)\equiv U_3^2,
\end{aligned} \tag{3}$$

$$\begin{aligned}
\dot{u}_1^{(3)} &= \frac{c^2-\left(u_1^{(3)}\right)^2}{c^2}\left(G_1^{(31)}+G_1^{(32)}+G_1^{(3)rad}\right)-\frac{u_1^{(3)}u_2^{(3)}}{c^2}\left(G_2^{(31)}+G_2^{(32)}+G_2^{(3)rad}\right)-\frac{u_1^{(3)}u_3^{(3)}}{c^2}\left(G_3^{(31)}+G_3^{(32)}+G_3^{(3)rad}\right)\equiv U_1^3\\
\dot{u}_2^{(3)} &= -\frac{u_1^{(3)}u_2^{(3)}}{c^2}\left(G_1^{(31)}+G_1^{(32)}+G_1^{(3)rad}\right)+\frac{c^2-(u_2^{(3)})^2}{c^2}\left(G_2^{(31)}+G_2^{(32)}+G_2^{(3)rad}\right)-\frac{u_2^{(3)}u_3^{(3)}}{c^2}\left(G_3^{(31)}+G_3^{(32)}+G_3^{(3)rad}\right)\equiv U_2^3\\
\dot{u}_3^{(3)} &= -\frac{u_1^{(3)}u_3^{(3)}}{c^2}\left(G_1^{(31)}+G_1^{(32)}+G_1^{(3)rad}\right)-\frac{u_2^{(3)}u_3^{(3)}}{c^2}\left(G_2^{(31)}+G_2^{(32)}+G_2^{(3)rad}\right)+\frac{c^2-(u_3^{(3)})^2}{c^2}\left(G_3^{(31)}+G_3^{(32)}+G_3^{(3)rad}\right)\equiv U_3^3,\ t\geq 0,
\end{aligned} \tag{4}$$

where

$$\begin{aligned}
G_\alpha^{(12)} = \frac{e_1e_2}{m_1}\frac{\Delta_1}{c^3}&\left\{\frac{\left(c^2-\left\langle\vec{u}^{(1)},\vec{u}^{(2)}\right\rangle\right)\xi_\alpha^{(12)}-\left(c^2\tau_{12}-\left\langle\vec{\xi}^{(12)},\vec{u}^{(1)}\right\rangle\right)u_\alpha^{(2)}}{\left(c^2\tau_{12}-\left\langle\vec{\xi}^{(12)},\vec{u}^{(2)}\right\rangle\right)^3}\,\frac{\Delta_{12}^4+D_{12}\left[\Delta_{12}^2\left\langle\vec{\xi}^{(12)},\dot{\vec{u}}^{(2)}\right\rangle+\left(\left\langle\vec{\xi}^{(12)},\vec{u}^{(2)}\right\rangle-c^2\tau_{12}\right)\left\langle u^{(2)},\dot{u}^{(2)}\right\rangle\right]}{\Delta_{12}^2}-\right.\\
&\left.-\frac{D_{12}}{\left(\left\langle\vec{\xi}^{(12)},\vec{u}^{(2)}\right\rangle-c^2\tau_{12}\right)^2}\,\frac{\left(\Delta_{12}^2\dot{u}_\alpha^{(2)}+u_\alpha^{(2)}\left\langle\vec{u}^{(2)},\dot{\vec{u}}^{(2)}\right\rangle\right)\left(c^2\tau_{12}-\left\langle\vec{\xi}^{(12)},\vec{u}^{(1)}\right\rangle\right)+\left(\Delta_{12}^2\left\langle\vec{u}^{(1)},\dot{\vec{u}}^{(2)}\right\rangle+\left(\left\langle\vec{u}^{(1)},\vec{u}^{(2)}\right\rangle-c^2\right)\left\langle\vec{u}^{(2)},\dot{\vec{u}}^{(2)}\right\rangle\right)\xi_\alpha^{(12)}}{\Delta_{12}^2}\right\};\\
G_\alpha^{(13)} = \frac{e_1e_3}{m_1}\frac{\Delta_1}{c^3}&\left\{\frac{\left(c^2-\left\langle\vec{u}^{(1)},\vec{u}^{(3)}\right\rangle\right)\xi_\alpha^{(13)}-\left(c^2\tau_{13}-\left\langle\vec{\xi}^{(13)},\vec{u}^{(1)}\right\rangle\right)u_\alpha^{(3)}}{\left(c^2\tau_{13}-\left\langle\vec{\xi}^{(13)},\vec{u}^{(3)}\right\rangle\right)^3}\,\frac{\Delta_{13}^4+D_{13}\left[\Delta_{13}^2\left\langle\vec{\xi}^{(13)},\dot{\vec{u}}^{(3)}\right\rangle+\left(\left\langle\xi^{(13)},\vec{u}^{(3)}\right\rangle-c^2\tau_{13}\right)\left\langle\vec{u}^{(3)},\dot{\vec{u}}^{(3)}\right\rangle\right]}{\Delta_{13}^2}-\right.\\
&\left.-\frac{D_{13}}{\left(\left\langle\vec{\xi}^{(13)},\vec{u}^{(3)}\right\rangle-c^2\tau_{13}\right)^2}\,\frac{\left(\Delta_{13}^2\dot{u}_\alpha^{(3)}+u_\alpha^{(3)}\left\langle\vec{u}^{(3)},\dot{\vec{u}}^{(3)}\right\rangle\right)\left(c^2\tau_{13}-\left\langle\vec{\xi}^{(13)},\vec{u}^{(1)}\right\rangle\right)+\left(\Delta_{13}^2\left\langle\vec{u}^{(1)},\dot{\vec{u}}^{(3)}\right\rangle+\left(\left\langle\vec{u}^{(1)},\vec{u}^{(3)}\right\rangle-c^2\right)\left\langle\vec{u}^{(3)},\dot{\vec{u}}^{(3)}\right\rangle\right)\xi_\alpha^{(13)}}{\Delta_{13}^2}\right\};\\
G_\alpha^{(21)} = \frac{e_2e_1}{m_2}\frac{\Delta_2}{c^3}&\left\{\frac{\left(c^2-\left\langle\vec{u}^{(2)},\vec{u}^{(1)}\right\rangle\right)\xi_\alpha^{(21)}-\left(c^2\tau_{21}-\left\langle\vec{\xi}^{(21)},\vec{u}^{(2)}\right\rangle\right)u_\alpha^{(1)}}{\left(c^2\tau_{21}-\left\langle\vec{\xi}^{(21)},\vec{u}^{(1)}\right\rangle\right)^3}\,\frac{\Delta_{21}^4+D_{21}\left[\Delta_{21}^2\left\langle\vec{\xi}^{(21)},\dot{\vec{u}}^{(1)}\right\rangle+\left(\left\langle\vec{\xi}^{(21)},\vec{u}^{(1)}\right\rangle-c^2\tau_{21}\right)\left\langle\vec{u}^{(1)},\dot{\vec{u}}^{(1)}\right\rangle\right]}{\Delta_{21}^2}-\right.\\
&\left.-\frac{D_{23}}{\left(\left\langle\vec{\xi}^{(21)},\vec{u}^{(1)}\right\rangle-c^2\tau_{21}\right)^2}\,\frac{\left(\Delta_{21}^2\dot{u}_\alpha^{(1)}+u_\alpha^{(1)}\left\langle\vec{u}^{(1)},\dot{\vec{u}}^{(1)}\right\rangle\right)\left(c^2\tau_{21}-\left\langle\vec{\xi}^{(21)},\vec{u}^{(1)}\right\rangle\right)+\left(\Delta_{21}^2\left\langle\vec{u}^{(2)},\dot{\vec{u}}^{(1)}\right\rangle+\left(\left\langle\vec{u}^{(2)},\vec{u}^{(1)}\right\rangle-c^2\right)\left\langle\vec{u}^{(1)},\dot{\vec{u}}^{(1)}\right\rangle\right)\xi_\alpha^{(21)}}{\Delta_{21}^2}\right\};\\
G_\alpha^{(23)} = \frac{e_2e_3}{m_2}\frac{\Delta_2}{c^3}&\left\{\frac{\left(c^2-\left\langle\vec{u}^{(2)},\vec{u}^{(3)}\right\rangle\right)\xi_\alpha^{(23)}-\left(c^2\tau_{23}-\left\langle\vec{\xi}^{(23)},\vec{u}^{(2)}\right\rangle\right)u_\alpha^{(3)}}{\left(c^2\tau_{23}-\left\langle\vec{\xi}^{(23)},\vec{u}^{(3)}\right\rangle\right)^3}\,\frac{\Delta_{23}^4+D_{23}\left[\Delta_{23}^2\left\langle\vec{\xi}^{(23)},\dot{\vec{u}}^{(3)}\right\rangle+\left(\left\langle\vec{\xi}^{(23)},\vec{u}^{(3)}\right\rangle-c^2\tau_{23}\right)\left\langle\vec{u}^{(3)},\dot{\vec{u}}^{(3)}\right\rangle\right]}{\Delta_{23}^2}-\right.\\
&\left.-\frac{D_{23}}{\left(\left\langle\vec{\xi}^{(23)},\vec{u}^{(3)}\right\rangle,-,c^2,\tau_{23}\right)^2}\,\frac{\left(\Delta_{23}^2\dot{u}_\alpha^{(3)}+u_\alpha^{(3)}\left\langle\vec{u}^{(3)},\dot{\vec{u}}^{(3)}\right\rangle\right)\left(c^2\tau_{23}-\left\langle\vec{\xi}^{(23)},\vec{u}^{(2)}\right\rangle\right)+\left(\Delta_{23}^2\left\langle\vec{u}^{(2)},\dot{\vec{u}}^{(3)}\right\rangle+\left(\left\langle\vec{u}^{(2)},\vec{u}^{(3)}\right\rangle-c^2\right)\left\langle\vec{u}^{(3)},\dot{\vec{u}}^{(3)}\right\rangle\right)\xi_\alpha^{(23)}}{\Delta_{23}^2}\right\};\\
G_\alpha^{(31)} = \frac{e_3e_1}{m_3}\frac{\Delta_3}{c^3}&\left\{\frac{\left(c^2-\left\langle\vec{u}^{(3)},\vec{u}^{(1)}\right\rangle\right)\xi_\alpha^{(31)}-\left(c^2\tau_{31}-\left\langle\vec{\xi}^{(31)},\vec{u}^{(3)}\right\rangle\right)u_\alpha^{(1)}}{\left(c^2\tau_{31}-\left\langle\vec{\xi}^{(31)},,,\vec{u}^{(1)}\right\rangle\right)^3}\,\frac{\Delta_{31}^4+D_{31}\left[\Delta_{31}^2\left\langle\vec{\xi}^{(31)},\dot{\vec{u}}^{(1)}\right\rangle+\left(\left\langle\vec{\xi}^{(31)},\vec{u}^{(1)}\right\rangle-c^2\tau_{31}\right)\left\langle\vec{u}^{(1)},\dot{\vec{u}}^{(1)}\right\rangle\right]}{\Delta_{31}^2}-\right.\\
&\left.-\frac{D_{31}}{\left(\left\langle\vec{\xi}^{(31)},\vec{u}^{(1)}\right\rangle-c^2\tau_{31}\right)^2}\,\frac{\left(\Delta_{31}^2\dot{u}_\alpha^{(1)}+u_\alpha^{(1)}\left\langle\vec{u}^{(1)},\dot{\vec{u}}^{(1)}\right\rangle\right)\left(c^2\tau_{31}-\left\langle\vec{\xi}^{(31)},\vec{u}^{(3)}\right\rangle\right)+\left(\Delta_{31}^2\left\langle\vec{u}^{(3)},\dot{\vec{u}}^{(1)}\right\rangle+\left(\left\langle\vec{u}^{(3)},\vec{u}^{(1)}\right\rangle-c^2\right)\left\langle\vec{u}^{(1)},\dot{\vec{u}}^{(1)}\right\rangle\right)\xi_\alpha^{(31)}}{\Delta_{31}^2}\right\};\\
G_\alpha^{(32)} = \frac{e_3e_2}{m_3}\frac{\Delta_3}{c^3}&\left\{\frac{\left(c^2-\left\langle\vec{u}^{(3)},\vec{u}^{(2)}\right\rangle\right)\xi_\alpha^{(32)}-\left(c^2\tau_{32}-\left\langle\vec{\xi}^{(32)},\vec{u}^{(3)}\right\rangle\right)u_\alpha^{(2)}}{\left(c^2\tau_{32}-\left\langle\vec{\xi}^{(32)},\vec{u}^{(2)}\right\rangle\right)^3}\,\frac{\Delta_{32}^4+D_{32}\left[\Delta_{32}^2\left\langle\vec{\xi}^{(32)},\dot{\vec{u}}^{(2)}\right\rangle+\left(\left\langle\vec{\xi}^{(32)},\vec{u}^{(2)}\right\rangle-c^2\tau_{32}\right)\left\langle\vec{u}^{(2)},\dot{\vec{u}}^{(2)}\right\rangle\right]}{\Delta_{32}^2}-\right.\\
&\left.-\frac{D_{32}}{\left(\left\langle\vec{\xi}^{(32)},\vec{u}^{(2)}\right\rangle-c^2\tau_{32}\right)^2}\,\frac{\left(\Delta_{32}^2\dot{u}_\alpha^{(2)}+u_\alpha^{(2)}\left\langle\vec{u}^{(2)},\dot{\vec{u}}^{(2)}\right\rangle\right)\left(c^2\tau_{32}-\left\langle\vec{\xi}^{(32)},\vec{u}^{(3)}\right\rangle\right)+\left(\Delta_{32}^2\left\langle\vec{u}^{(3)},\dot{\vec{u}}^{(2)}\right\rangle+\left(\left\langle\vec{u}^{(3)},\vec{u}^{(2)}\right\rangle-c^2\right)\left\langle\vec{u}^{(2)},\dot{\vec{u}}^{(2)}\right\rangle\right)\xi_\alpha^{(32)}}{\Delta_{32}^2}\right\};
\end{aligned}$$

$$G_\alpha^{(k)rad} = -\frac{e_k^2}{m_kc^2}\left(\frac{u_\alpha^{(k)}(t)}{{\Delta_k}^3}\left\langle\vec{u}^{(k)}(t),\ddot{\vec{u}}^{(k)}(t)\right\rangle+\frac{\ddot{u}_\alpha^{(k)}(t)}{\Delta_k}\right),(k=1,2,3), \tag{5}$$

where $D_{km} = \dfrac{c\sqrt{\left\langle \vec{\xi}^{(km)}, \vec{\xi}^{(km)} \right\rangle} - \left\langle \vec{\xi}^{(km)}, \vec{u}^{(m)} \right\rangle}{c\sqrt{\left\langle \vec{\xi}^{(km)}, \vec{\xi}^{(km)} \right\rangle} - \left\langle \vec{\xi}^{(km)}, \vec{u}^{(k)} \right\rangle}$ for $m \neq k$.

In accordance with the Kepler formulation, the first particle P_1 is at the origin $O(0,0,0)$, and therefore,

$$x_\alpha^{(1)}(t) = 0,\ u_\alpha^{(1)}(t) = 0,\ \dot{u}_\alpha^{(1)}(t) = 0\ (\alpha = 1,2,3). \tag{6}$$

Then, we obtain $0 = G_1^{(12)} + G_1^{(13)}$, $0 = G_2^{(12)} + G_2^{(13)}$, $0 = G_3^{(12)} + G_3^{(13)}$.

It is easy to see that the equation $0 = G_1^{(12)} + G_1^{(13)}$ becomes $0 = 0$.

Remark 1. *We notice that in the equation $0 = G_2^{(12)} + G_2^{(13)}$, the unknown functions and their derivatives have retarded arguments $t - \tau_{12}$, while in the equation $0 = G_3^{(12)} + G_3^{(13)}$, the arguments are $t - \tau_{13}$. This implies that we must consider these equations on the initial interval $[-T, 0]$. This is why we call the first group of three equations (2) initial equations. We consider the second and third group of Equations (3) and (4), respectively (six in number), on the interval $[0, \infty)$.*

To simplify the equations of motion, we notice that for the first Bohr orbit, the Sommerfeld fine structure constant is $\beta = \bar{c}/c \approx 1/137 \Rightarrow \beta^2 \approx 0$. Consequently, $\Delta_k = \sqrt{c^2 - \left\langle \vec{u}^{(k)}, \vec{u}^{(k)} \right\rangle} = c\sqrt{1 - \left\langle \vec{u}^{(k)}, \vec{u}^{(k)} \right\rangle / c^2} \approx c$, $\Delta_{km} = \sqrt{c^2 - \left\langle \vec{u}^{(m)}(t - \tau_{km}), \vec{u}^{(m)}(t - \tau_{km}) \right\rangle} \approx c$, and $c^2\left(1 - \dfrac{\left\langle \vec{u}^{(2)}, \vec{u}^{(3)} \right\rangle}{c^2}\right) \approx c^2$.

If ρ_{km} is the distance between the k-th and m-th particle via $\xi^{(km)} \approx \rho_{km} \approx 10^{-11}$ m (cf. [14]), we obtain

$$c^2\tau_{23} - \left\langle \vec{\xi}^{(23)}, \vec{u}^{(2)} \right\rangle \approx c^2\tau_{23},\ D_{23} = \frac{c^2\tau_{23} - \left\langle \vec{\xi}^{(23)}, \vec{u}^{(3)} \right\rangle}{c^2\tau_{23} - \left\langle \vec{\xi}^{(23)}, \vec{u}^{(2)} \right\rangle} \approx 1 \text{ and}$$

$$H_{23} = \frac{\Delta_{23}^4 + D_{23}\left[\Delta_{23}^2 \left\langle \vec{\xi}^{(23)}, \dot{\vec{u}}^{(3)} \right\rangle - \left(c^2\tau_{23} - \left\langle \vec{\xi}^{(23)}, \vec{u}^{(3)} \right\rangle\right)\left\langle \vec{u}^{(3)}, \dot{\vec{u}}^{(3)} \right\rangle\right]}{\Delta_{23}^2} \approx c^2 + \left\langle \vec{\xi}^{(23)} - \tau_{23}\vec{u}^{(3)}, \dot{\vec{u}}^{(3)} \right\rangle.$$

We can simplify the above equations in accordance to the last relations:

$$G_\alpha^{(12)} = \frac{e_1 e_2}{m_1}\frac{1}{c^6}\frac{\left(c^2 + \left\langle \vec{\xi}^{(12)}, \dot{\vec{u}}^{(2)} \right\rangle\right)\left(\xi_\alpha^{(12)} - \tau_{12}u_\alpha^{(2)}\right) - c^2{\tau_{12}}^2\dot{u}_\alpha^{(2)}}{{\tau_{12}}^3};$$

$$G_\alpha^{(13)} = \frac{e_1 e_3}{m_1}\frac{1}{c^6}\frac{\left(c^2 + \left\langle \vec{\xi}^{(13)}, \dot{\vec{u}}^{(3)} \right\rangle\right)\left(\xi_\alpha^{(13)} - \tau_{13}u_\alpha^{(3)}\right) - c^2{\tau_{13}}^2\dot{u}_\alpha^{(3)}}{{\tau_{13}}^3};$$

$$G_\alpha^{(21)} = \frac{e_2 e_1}{m_2}\frac{\Delta_2}{c^3}\frac{c^2\xi_\alpha^{(21)}}{(c^2\tau_{21})^3}\frac{\Delta_{21}^4}{\Delta_{21}^2} \approx \frac{e_2 e_1}{m_2}\frac{\xi_\alpha^{(21)}}{c^4{\tau_{21}}^3};\ G_\alpha^{(23)} = \frac{e_2 e_3}{m_2 c^3}\left(\frac{\xi_\alpha^{(23)}}{{\tau_{23}}^3}\frac{\left\langle \vec{\xi}^{(23)}, \dot{\vec{u}}^{(3)} \right\rangle}{c^3} - \frac{\dot{u}_\alpha^{(3)}}{c\tau_{23}}\right);$$

$$G_\alpha^{(31)} = \frac{e_3 e_1}{m_3}\frac{\Delta_3}{c^3}\frac{c^2\xi_\alpha^{(31)}}{(c^2\tau_{31})^3}\frac{\Delta_{31}^4}{\Delta_{31}^2} \approx \frac{e_3 e_1}{m_3}\frac{\xi_\alpha^{(31)}}{c^4{\tau_{31}}^3};\ G_\alpha^{(32)} = \frac{e_2 e_3}{m_3 c^3}\left(\frac{\xi_\alpha^{(32)}}{{\tau_{32}}^3}\frac{\left\langle \vec{\xi}^{(32)}, \dot{\vec{u}}^{(3)} \right\rangle}{c^3} - \frac{\dot{u}_\alpha^{(2)}}{c\tau_{32}}\right);$$

$$G_\alpha^{(2)rad} = -\frac{e_2^2}{m_2 c^2}\left(\frac{u_\alpha^{(2)}\left\langle \vec{u}^{(2)}, \ddot{\vec{u}}^{(2)} \right\rangle}{c^3} + \frac{\ddot{u}_\alpha^{(2)}}{c}\right);\ G_\alpha^{(3)rad} = -\frac{e_3^2}{m_3 c^2}\left(\frac{u_\alpha^{(3)}\left\langle \vec{u}^{(3)}, \ddot{\vec{u}}^{(3)} \right\rangle}{c^3} + \frac{\ddot{u}_\alpha^{(3)}}{c}\right).$$

For the stationary particle, we have $G_\alpha^{(1)rad} = 0$. Therefore, the initial (vector) equations, $G_1^{(12)} + G_1^{(13)} = 0, G_2^{(12)} + G_2^{(13)} = 0$, $G_3^{(12)} + G_3^{(13)} = 0$, become:

$$\begin{aligned}
&\tfrac{e_1e_2}{m_1}\left[\left(c^2+\left\langle \vec{\xi}^{(12)}, \dot{\vec{u}}^{(2)}\right\rangle\right)\left(\xi_1^{(12)}-\tau_{12}u_1^{(2)}\right)-c^2\tau_{12}{}^2\dot{u}_1^{(2)}\right]+\tfrac{e_3e_1}{m_1}\left[\left(c^2+\left\langle \vec{\xi}^{(13)}, \dot{\vec{u}}^{(3)}\right\rangle\right)\left(\xi_1^{(13)}-\tau_{13}u_1^{(3)}\right)-c^2\tau_{13}{}^2\dot{u}_1^{(3)}\right]=0\\
&\tfrac{e_1e_2}{m_2}\left[\left(c^2+\left\langle \vec{\xi}^{(12)}, \dot{\vec{u}}^{(2)}\right\rangle\right)\left(\xi_2^{(12)}-\tau_{12}u_2^{(2)}\right)-c^2\tau_{12}{}^2\dot{u}_2^{(2)}\right]+\tfrac{e_3e_1}{m_2}\left[\left(c^2+\left\langle \vec{\xi}^{(13)}, \dot{\vec{u}}^{(3)}\right\rangle\right)\left(\xi_2^{(13)}-\tau_{13}u_2^{(3)}\right)-c^2\tau_{13}{}^2\dot{u}_2^{(3)}\right]=0,\\
&\tfrac{e_1e_2}{m_3}\left[\left(c^2+\left\langle \vec{\xi}^{(12)}, \dot{\vec{u}}^{(2)}\right\rangle\right)\left(\xi_3^{(12)}-\tau_{12}u_3^{(2)}\right)-c^2\tau_{12}{}^2\dot{u}_3^{(2)}\right]+\tfrac{e_3e_1}{m_3}\left[\left(c^2+\left\langle \vec{\xi}^{(13)}, \dot{\vec{u}}^{(3)}\right\rangle\right)\left(\xi_3^{(13)}-\tau_{13}u_3^{(3)}\right)-c^2\tau_{13}{}^2\dot{u}_3^{(3)}\right]=0.
\end{aligned} \tag{7}$$

We assume that

$$\begin{aligned}
&\left(c^2+\left\langle \vec{\xi}^{(12)}, \dot{\vec{u}}^{(2)}\right\rangle\right)\left(\xi_1^{(12)}-\tau_{12}u_1^{(2)}\right)-c^2\tau_{12}{}^2\dot{u}_1^{(2)}=0,\\
&\left(c^2+\left\langle \vec{\xi}^{(12)}, \dot{\vec{u}}^{(2)}\right\rangle\right)\left(\xi_2^{(12)}-\tau_{12}u_2^{(2)}\right)-c^2\tau_{12}{}^2\dot{u}_2^{(2)}=0,\\
&\left(c^2+\left\langle \vec{\xi}^{(12)}, \dot{\vec{u}}^{(2)}\right\rangle\right)\left(\xi_3^{(12)}-\tau_{12}u_3^{(2)}\right)-c^2\tau_{12}{}^2\dot{u}_3^{(2)}=0.
\end{aligned} \tag{8}$$

and

$$\begin{aligned}
&\left(c^2+\left\langle \vec{\xi}^{(13)}, \dot{\vec{u}}^{(3)}\right\rangle\right)\left(\xi_1^{(13)}-\tau_{13}u_1^{(3)}\right)-c^2\tau_{13}{}^2\dot{u}_1^{(3)}=0\\
&\left(c^2+\left\langle \vec{\xi}^{(13)}, \dot{\vec{u}}^{(3)}\right\rangle\right)\left(\xi_2^{(13)}-\tau_{13}u_2^{(3)}\right)-c^2\tau_{13}{}^2\dot{u}_2^{(3)}=0,\\
&\left(c^2+\left\langle \vec{\xi}^{(13)}, \dot{\vec{u}}^{(3)}\right\rangle\right)\left(\xi_3^{(13)}-\tau_{13}u_3^{(3)}\right)-c^2\tau_{13}{}^2\dot{u}_3^{(3)}=0.
\end{aligned} \tag{9}$$

The system (8) contains functions with the argument $t-\tau_{12}(t) \in [-T,0]$ for $t \in [0,T]$. The system (9) contains functions with the argument $t-\tau_{13}(t) \in [-T,0]$ for $t \in [0,T]$. If (8) and (9) are satisfied, then (7) is also satisfied.

Remark 2. *The choice of* $t-\tau_{km}(t) \in [-T,0]$ *follows from the next reasoning: Let T be the period of the solution. We notice that all the arguments of the unknown functions are* $t-\tau_{km}$*, that is,* $\vec{u} = \vec{u}^{(m)}(t-\tau_{km})$*. Therefore, we must look for a solution on the initial set, that is, for* $t-\tau_{km}(t) \in [\tau_0;\ 0]$*, where* $\tau_0 = \min\{t-\tau_{km}(t) : t \in [0,T]\}$*. Since* $1 - d\tau_{km}(t)/dt > 0$*, then* $t-\tau_{km}(t)$ *is an increasing function. We have proved that if the trajectories are T-periodic, then* $\tau_{km}(t)$ *is T-periodic too.*

It follows that $-T-\tau_{km}(-T) \le 0-\tau_{km}(0) \Leftrightarrow -T-\tau_{km}(0) \le -\tau_{km}(0) \Leftrightarrow \tau_{km}(0) \le T-\tau_{km}(0)$ and then $0-\tau_{km}(0) \le t-\tau_{km}(t) \le T-\tau_{km}(T) = T-\tau_{km}(0) = 0$. Consequently, $-T \le t-\tau_{km}(t) \le 0$. Therefore, the functions $t-\tau_{km}(t) : [0,T] \to [-T,0]$, which means that $\tau_0 = -T$, and we put $t-\tau_{km}(t) = \theta \in [-T,0]$. The arguments of the unknown functions $\vec{u}^{(2)} = \vec{u}^{(2)}(\theta)$, $\vec{u}^{(3)} = \vec{u}^{(3)}(\theta)$ belong to $[-T,0]$.

After the transformations given in Appendix A, we obtain from (8) and (9) the *initial equations*:

$$\begin{aligned}
&\ddot{\rho}_2(t-\tau_{23}) = \rho_2(t-\tau_{23})\dot{\varphi}_2{}^2(t-\tau_{23})\cos^2\lambda_2(t-\tau_{23}) + \rho_2(t-\tau_{23})\dot{\lambda}_2{}^2(t-\tau_{23}) - \tfrac{c^2}{\rho_2(t-\tau_{23})},\\
&\ddot{\varphi}_2(t-\tau_{23}) = 2\dot{\varphi}_2(t-\tau_{23})\dot{\lambda}_2(t-\tau_{23})tg\lambda_2(t-\tau_{23}) - \tfrac{2\dot{\rho}_2(t-\tau_{23})\dot{\varphi}_2(t-\tau_{23})}{\rho_n(t-\tau_{23})},\\
&\ddot{\lambda}_2(t-\tau_{23}) = -\tfrac{2\dot{\rho}_2(t-\tau_{23})\dot{\lambda}_2(t-\tau_{23}) + \rho_2(t-\tau_{23})\dot{\varphi}_2{}^2(t-\tau_{23})\sin\lambda_2(t-\tau_{23})\ \cos\lambda_2(t-\tau_{23})}{\rho_2(t-\tau_{23})}.
\end{aligned} \tag{10-1}$$

for the second particle and

$$\begin{aligned}
&\ddot{\rho}_3(t-\tau_{32}) = \rho_3(t-\tau_{32})\dot{\varphi}_3{}^2(t-\tau_{32})\cos^2\lambda_3(t-\tau_{32}) + \rho_3(t-\tau_{32})\dot{\lambda}_3{}^2(t-\tau_{32}) - \tfrac{c^2}{\rho_3(t-\tau_{32})},\\
&\ddot{\varphi}_3(t-\tau_{32}) = 2\dot{\varphi}_3(t-\tau_{32})\dot{\lambda}_3(t-\tau_{32})tg\lambda_3(t-\tau_{32}) - \tfrac{2\dot{\rho}_3(t-\tau_{32})\dot{\varphi}_3(t-\tau_{32})}{\rho_3(t-\tau_{32})},\\
&\ddot{\lambda}_3(t-\tau_{32}) = -\tfrac{2\dot{\rho}_3(t-\tau_{32})\dot{\lambda}_3(t-\tau_{32})+\rho_3(t-\tau_{32})\dot{\varphi}_3{}^2(t-\tau_{32})\sin\lambda_3(t-\tau_{32})\,\cos\lambda_3(t-\tau_{32})}{\rho_3(t-\tau_{32})}.
\end{aligned} \tag{10-2}$$

for the third particle.

The reasoning from Remark 2 implies that we look for a solution (10-1) and (10-2) on the interval $[-T,0]$.

The existence–uniqueness result for $(10-n)$ (n = 1, 2) can be obtained as in [19] in the functional space $M_{r_2}{}^0 \times M_{\phi_2}{}^0 \times M_{\eta_2}{}^0 \times M_{r_3}{}^0 \times M_{\phi_3}{}^0 \times M_{\eta_3}{}^0$, where

$$\begin{aligned}
M_{r_2}{}^0 &= \left\{r_{20}(.) \in C_T{}^\infty[-T,0] : \left|r_{20}{}^{(m)}(t)\right| \le \omega_2{}^m R_2 e^{\mu(t-T_k)};\ r_{20}(-T) = r_{20}(0) = 0\right\},\\
M_{\phi_2}{}^0 &= \left\{\phi_{20}(.) \in C_T{}^\infty[-T,0] : \left|\phi_{20}{}^{(m)}(t)\right| \le \omega_2{}^m \Phi_2 e^{\mu(t-T_k)},\ \phi_{20}(-T) = \phi_{20}(0) = 0\right\},\\
M_{\eta_2}{}^0 &= \left\{\eta_{20}(.) \in C_T{}^\infty[-T,0] : \left|\eta_2{}^{(m)}(t)\right| \le \omega_2{}^m \Phi_2 e^{\mu(t-T_k)},\ \eta_{20}(-T) = \eta_{20}(0) = 0\right\},\\
M_{r_3}{}^0 &= \left\{r_{30}(.) \in C_T{}^\infty[-T,0] : \left|r_{30}{}^{(m)}(t)\right| \le \omega_3{}^m R_3 e^{\mu(t-T_k)};\ r_{30}(-T) = r_{30}(0) = 0\right\},\\
M_{\phi_3}{}^0 &= \left\{\phi_{30} \in C_T{}^\infty[-T,0] : \left|\phi_{30}{}^{(m)}(t)\right| \le \omega_3{}^m \Phi_3 e^{\mu(t-T_k)},\ \phi_{30}(-T) = \phi_{30}(0) = 0\right\},\\
M_{\eta_3}{}^0 &= \left\{\eta_{30} \in C_T{}^\infty[-T,0] : \left|\eta_{30}{}^{(m)}(t)\right| \le \omega_3{}^m \Upsilon_3 e^{\mu(t-T_k)},\ \eta_{30}(-T) = \eta_{30}(0) = 0\right\}.
\end{aligned}$$

as fixed points of the operators

$$\begin{aligned}
B_{r_2}(r_2,\phi_2,\eta_2,r_3,\phi_3,\eta_3)(t) &:= \int_{-T}^{t} G_{r_2}(s)ds - \tfrac{t+T}{T}\int_{-T}^{0} G_{r_2}(s)ds,\ t \in [-T,0],\\
B_{r_3}(r_2,\phi_2,\eta_2,r_3,\phi_3,\eta_3)(t) &:= \int_{-T}^{t} G_{r_3}(s)ds - \tfrac{t+T}{T}\int_{-T}^{0} G_{r_3}(s)ds,\ t \in [-T,0],\\
B_{\phi_2}(r_2,\phi_2,\eta_2,r_3,\phi_3,\eta_3)(t) &:= \int_{-T}^{t} G_{\phi_2}(s)ds - \tfrac{t+T}{T}\int_{-T}^{0} G_{\phi_2}(s)ds,\ t \in [-T,0],\\
B_{\phi_3}(r_2,\phi_2,\eta_2,r_3,\phi_3,\eta_3)(t) &:= \int_{-T}^{t} G_{r_3}(s)ds - \tfrac{t+T}{T}\int_{-T}^{0} G_{r_3}(s)ds,\ t \in [-T,0],\\
B_{\eta_2}(r_2,\phi_2,\eta_2,r_3,\phi_3,\eta_3)(t) &:= \int_{-T}^{t} G_{\eta_2}(s)ds - \tfrac{t+T}{T}\int_{-T}^{0} G_{\eta_2}(s)ds,\ t \in [-T,0],\\
B_{\eta_3}(r_2,\phi_2,\eta_2,r_3,\phi_3,\eta_3)(t) &:= \int_{-T}^{t} G_{\eta_3}(s)ds - \tfrac{t+T}{T}\int_{-T}^{0} G_{\eta_3}(s)ds,\ t \in [-T,0].
\end{aligned}$$

It is easy to verify that

$$B_{r_2}(.)(-T) = \int_{-T}^{-T} G_{r_2}(s)ds - \frac{-T+T}{T}\int_{-T}^{0} G_{r_2}(s)ds = 0 = \int_{-T}^{0} G_{r_2}(s)ds - \frac{T}{T}\int_{-T}^{0} G_{r_2}(s)ds = B_{r_2}(.)(0)$$

and so on. Then, we take the solutions as the initial functions $r_{20}(t), \phi_{20}(t), \eta_{20}(t), r_{30}(t), \phi_{30}(t), \eta_{30}(t)$ for the basic equations below.

3. Basic Equations in Spherical Coordinates

We note that $\varphi_{23} = \varphi_2(t) - \varphi_3(t-\tau_{23})$, $\varphi_{32} = \varphi_3(t) - \varphi_2(t-\tau_{32})$.

From (2)–(4), one obtains, in Appendix B, the final form of the equations of motion of the second and third particles:

$$\ddot{\rho}_2(t) = \frac{e_2e_1}{m_2}\frac{1}{c\rho_2{}^2(t)} - \frac{e_2e_3}{m_2}\frac{\rho_2(t)-\rho_3(t-\tau_{23})(\cos\varphi_{23}\cos\lambda_2(t)\cos\lambda_3(t-\tau_{23})+\sin\lambda_2(t)\sin\lambda_3(t-\tau_{23}))}{c^3}\frac{\left\langle \vec{\xi}^{(23)},\dot{\vec{u}}^{(3)} \right\rangle}{c^3\tau_{23}{}^3} -$$

$$-\frac{e_2e_3}{m_2}\frac{\dot{\rho}_3(\cos\varphi_{23}\cos\lambda_2(t)\cos\lambda_3(t-\tau_{23})+\sin\lambda_2(t)\sin\lambda_3(t-\tau_{23}))+\rho_3(t-\tau_{23})\dot{\varphi}_3(t-\tau_{23})\sin\varphi_{23}\cos\lambda_2(t)\cos\lambda_3(t-\tau_{23})}{c^4\tau_{23}} -$$

$$-\frac{e_2e_3}{m_2}\frac{\rho_3\dot{\lambda}_3(\sin\lambda_2(t)\cos\lambda_3(t-\tau_{23})-\cos\varphi_{23}\sin\lambda_3(t-\tau_{23})\cos\lambda_2(t))}{c^4\tau_{23}} - \frac{e_2^2}{m_2}\frac{\dddot{\rho}_2(t)}{c^3};$$

$$\ddot{\varphi}_2(t) = \frac{e_2e_3}{m_2}\left[\frac{\rho_3(t-\tau_{23})\sin\varphi_{23}\cos\lambda_3(t-\tau_{23})}{\tau_{23}{}^3\rho_2(t)\cos\lambda_2(t)}\frac{\left\langle \vec{\xi}^{(23)},\dot{\vec{u}}^{(3)} \right\rangle}{c^6} +\right.$$

$$+\frac{\dot{\rho}_3(t-\tau_{23})\cos\lambda_3(t-\tau_{23})\sin\varphi_{23}+\rho_3(t-\tau_{23})\dot{\varphi}_3(t-\tau_{23})\cos\lambda_3(t-\tau_{23})\cos\varphi_{23}}{c^4\tau_{23}\rho_2(t)\cos\lambda_2(t)} -$$

$$\left.-\frac{\rho_3(t-\tau_{23})\dot{\lambda}_3(t-\tau_{23})\sin\lambda_3(t-\tau_{23})\sin\varphi_{23}}{c^4\tau_{23}\rho_2(t)\cos\lambda_2(t)} - \frac{\dddot{\varphi}_2(t)}{c^3}\right];$$

$$\ddot{\lambda}_2(t) = \frac{e_2e_3}{m_2}\frac{\rho_3(t-\tau_{23})\cos\varphi_{23}\sin\lambda_2(t)\cos\lambda_3(t-\tau_{23})}{c^3\tau_{23}{}^3}\frac{\left\langle \vec{\xi}^{(23)},\dot{\vec{u}}^{(3)} \right\rangle}{c^3\rho_2} +$$

$$+\frac{e_2e_3}{m_2}\frac{\dot{\rho}_3(\cos\varphi_{23}\cos\lambda_3(t-\tau_{23})\sin\lambda_2(t)-\sin\lambda_3(t-\tau_{23})\cos\lambda_2(t))+\rho_3\dot{\varphi}_3\sin\varphi_{23}\sin\lambda_2(t)\cos\lambda_3(t-\tau_{23})}{c^4\tau_{23}\rho_2(t)} +$$

$$+\frac{e_2e_3}{m_2}\frac{-\rho_3\dot{\lambda}_3(\cos\varphi_{23}\sin\lambda_2(t)\sin\lambda_3(t-\tau_{23})+\cos\lambda_2(t)\cos\lambda_3(t-\tau_{23})}{c^4\tau_{23}\rho_2(t)} - \frac{e_2^2}{m_2}\frac{\dddot{\lambda}_2(t)}{c^3};$$

where

$$\ddot{\rho}_3(t) = \frac{e_3e_1}{m_3}\frac{1}{c\rho_3{}^2(t)} - \frac{e_2e_3}{m_3}\frac{\rho_3(t)-\rho_2(t-\tau_{32})(\cos\varphi_{32}\cos\lambda_2(t-\tau_{32})\cos\lambda_3(t)+\sin\lambda_2(t-\tau_{32})\sin\lambda_3(t))}{c^3}\frac{\left\langle \vec{\xi}^{(32)},\dot{\vec{u}}^{(2)} \right\rangle}{c^3\tau_{32}{}^3}$$

$$-\frac{e_3e_2}{m_3}\frac{\dot{\rho}_2(\cos\varphi_{32}\cos\lambda_2(t-\tau_{32})\cos\lambda_3(t)+\sin\lambda_2(t-\tau_{32})\sin\lambda_3(t))+\rho_2(t-\tau_{32})\dot{\varphi}_2(t-\tau_{32})\sin\varphi_{32}\cos\lambda_2(t-\tau_{32})\cos\lambda_3(t)}{c^4\tau_{23}}$$

$$-\frac{e_3e_2}{m_3}\frac{\rho_2(t-\tau_{32})\dot{\lambda}_2(t-\tau_{32})(\sin\lambda_3(t)\cos\lambda_2(t-\tau_{32})-\cos\varphi_{32}\sin\lambda_2(t-\tau_{32})\cos\lambda_3(t))}{c^4\tau_{23}} - \frac{e_3^2}{m_3}\frac{\dddot{\rho}_3(t)}{c^3};$$

$$\ddot{\varphi}_3 = \frac{e_3e_2}{m_3}\left[\frac{\rho_2(t-\tau_{32})\sin\varphi_{32}\cos\lambda_2(t-\tau_{32})}{\tau_{32}{}^3\rho_3(t)\cos\lambda_3(t)}\frac{\left\langle \vec{\xi}^{(32)},\dot{\vec{u}}^{(2)} \right\rangle}{c^6} +\right.$$

$$+\frac{\dot{\rho}_2(t-\tau_{32})\cos\lambda_2(t-\tau_{32})\sin\varphi_{32}+\rho_2(t-\tau_{32})\dot{\varphi}_2(t-\tau_{32})\cos\lambda_2(t-\tau_{32})\cos\varphi_{32}}{c^4\tau_{32}\rho_3\cos\lambda_3(t)} -$$

$$\left.-\frac{\rho_2(t-\tau_{32})\dot{\lambda}_2(t-\tau_{32})\sin\lambda_2(t-\tau_{32})\sin\varphi_{32}}{c^4\tau_{32}\rho_3\cos\lambda_3(t)} - \frac{\dddot{\varphi}_3(t)}{c^3}\right];$$

$$\ddot{\lambda}_3(t) = \frac{e_3e_2}{m_3}\frac{\rho_2(t-\tau_{32})\cos\varphi_{32}\sin\lambda_3(t)\cos\lambda_2(t-\tau_{32})}{c^3\tau_{32}{}^3}\frac{\left\langle \vec{\xi}^{(32)},\dot{\vec{u}}^{(2)} \right\rangle}{c^3\rho_3(t)} +$$

$$+\frac{e_3e_2}{m_3c^3}\frac{\dot{\rho}_3(\cos\varphi_{32}\cos\lambda_2(t-\tau_{32})\sin\lambda_3(t)-\sin\lambda_2(t-\tau_{32})\cos\lambda_3(t))+\rho_2(t-\tau_{32})\dot{\varphi}_2(t-\tau_{32})\sin\varphi_{32}\sin\lambda_3(t)\cos\lambda_2(t-\tau_{32})}{c\tau_{32}\rho_3(t)}$$

$$+\frac{e_3e_2}{m_3c^3}\frac{-\rho_2(t-\tau_{32})\dot{\lambda}_2(t-\tau_{32})(\cos\varphi_{23}\sin\lambda_2(t-\tau_{32})\sin\lambda_3(t)+\cos\lambda_2(t-\tau_{32})\cos\lambda_3(t)}{c\tau_{32}\rho_3(t)} - \frac{e_3^2}{m_3}\frac{\dddot{\lambda}_3(t)}{c^3}.$$

and

$$\left\langle \vec{\xi}^{(32)}, \dot{\vec{u}}^{(2)} \right\rangle =$$
$$= [\rho_3(t)\cos(\varphi_3(t) - \varphi_2(t-\tau_{32})\cos\lambda_3(t)\cos\lambda_2(t-\tau_{32}) + \rho_3(t)\sin\lambda_3(t)\sin\lambda_2(t-\tau_{32}) - \rho_2(t-\tau_{32})]\ddot{\rho}_2(t-\tau_{32}) +$$
$$+ \sin(\varphi_3(t) - \varphi_2(t-\tau_{32}))\cos\lambda_3(t)\cos\lambda_2(t-\tau_{32})\rho_2(t-\tau_{32})\rho_3(t)\ddot{\varphi}_2(t-\tau_{32}) +$$
$$+ [\sin\lambda_3(t)\cos\lambda_2(t-\tau_{32}) - \cos(\varphi_3(t) - \varphi_2(t-\tau_{32}))\sin\lambda_2(t-\tau_{32})\cos\lambda_3(t)]\rho_2(t-\tau_{32})\rho_3(t)\ddot{\lambda}_2(t-\tau_{32}).$$

After the substitution,

$$\dot{\rho}_n = r_n,\ \dot{\varphi}_n = \phi_n, \dot{\lambda}_n = \eta_n\ (n = 2,3) \Rightarrow \rho_n(t) = \rho_{n0} + \int_0^t r_n(s)ds,\ \varphi_n = \varphi_{n0} + \int_0^t \phi_n(s)ds,\ \lambda_n = \lambda_{n0} + \int_0^t \eta_n(s)ds$$

the system of equations of motion of the second and third particles can be rewritten as

$$\dot{r}_2 = \frac{e_2 e_1}{m_2}\frac{1}{c\rho_2{}^2} - \frac{e_2 e_3}{m_2}\frac{\rho_2 - \rho_3(\cos\varphi_{23}\cos\lambda_2\cos\lambda_3 + \sin\lambda_2\sin\lambda_3)}{c^3}\frac{\left\langle \vec{\xi}^{(23)}, \dot{\vec{u}}^{(3)} \right\rangle}{c^3\tau_{23}{}^3}$$
$$-\frac{e_2 e_3}{m_2}\frac{\dot{r}_3(\cos\varphi_{23}\cos\lambda_2\cos\lambda_3 + \sin\lambda_2\sin\lambda_3) + \rho_3\dot{\phi}_3\sin\varphi_{23}\cos\lambda_2\cos\lambda_3 + \rho_3\dot{\eta}_3(\sin\lambda_2\cos\lambda_3 - \cos\varphi_{23}\sin\lambda_3\cos\lambda_2)}{c^4\tau_{23}} - \frac{e_2^2}{m_2}\frac{\ddot{r}_2}{c^3} \equiv G_{r_2};$$
$$\dot{\phi}_2 = \frac{e_2 e_3}{m_2}\left[\frac{\rho_3\sin\varphi_{23}\cos\lambda_3}{\tau_{23}{}^3\rho_2\cos\lambda_2}\frac{\left\langle \vec{\xi}^{(23)}, \dot{\vec{u}}^{(3)} \right\rangle}{c^6} + \frac{\dot{r}_3\cos\lambda_3\sin\varphi_{23} + \rho_3\dot{\phi}_3\cos_3\cos\varphi_{23} - \rho_3\dot{\eta}_3\sin\lambda_3\sin\varphi_{23}}{c^4\tau_{23}\rho_2\cos\lambda_2} - \frac{\ddot{\phi}_2}{c^3}\right] \equiv G_{\phi_2};$$
$$\dot{\eta}_2 = \frac{e_2 e_3}{m_2}\frac{\rho_3\cos\varphi_{23}\sin\lambda_2\cos\lambda_3}{c^3\tau_{23}{}^3}\frac{\left\langle \vec{\xi}^{(23)}, \dot{\vec{u}}^{(3)} \right\rangle}{c^3\rho_2} +$$
$$+\frac{e_2 e_3}{m_2}\frac{\dot{r}_3(\cos\varphi_{23}\cos\lambda_3\sin\lambda_2 - \sin\lambda_3\cos\lambda_2) + \rho_3\dot{\phi}_3\sin\varphi_{23}\sin\lambda_2\cos\lambda_3 - \rho_3\dot{\eta}_3(\cos\varphi_{23}\sin\lambda_2\sin\lambda_3 + \cos\lambda_2\cos\lambda_3}{c^4\tau_{23}\rho_2} - \frac{e_2^2}{m_2}\frac{\ddot{\eta}_2}{c^3} \equiv G_{\eta_2};$$
$$\dot{r}_3 = \frac{e_3 e_1}{m_3}\frac{1}{c\rho_3{}^2} - \frac{e_2 e_3}{m_3}\frac{\rho_3 - \rho_2(\cos\varphi_{32}\cos\lambda_2\cos\lambda_3 + \sin\lambda_2\sin\lambda_3)}{c^3}\frac{\left\langle \vec{\xi}^{(32)}, \dot{\vec{u}}^{(2)} \right\rangle}{c^3\tau_{32}{}^3}$$
$$-\frac{e_3 e_2}{m_3}\frac{\dot{r}_2(\cos\varphi_{32}\cos\lambda_2\cos\lambda_3 + \sin\lambda_2\sin\lambda_3) + \rho_2\dot{\phi}_2\sin\varphi_{32}\cos\lambda_2\cos\lambda_3 + \rho_2\dot{\eta}_2(\sin\lambda_3\cos\lambda_2 - \cos\varphi_{32}\sin\lambda_2\cos\lambda_3)}{c^4\tau_{23}} - \frac{e_3^2}{m_3}\frac{\ddot{r}_3}{c^3} \equiv G_{r_3};$$
$$\dot{\phi}_3 = \frac{e_3 e_2}{m_3}\left[\frac{\rho_2\sin\varphi_{32}\cos\lambda_2}{\tau_{32}{}^3\rho_3\cos\lambda_3}\frac{\left\langle \vec{\xi}^{(32)}, \dot{\vec{u}}^{(2)} \right\rangle}{c^6} + \frac{\dot{r}_2\cos\lambda_2\sin\varphi_{32} + \rho_2\dot{\phi}_2\cos_2\cos\varphi_{32} - \rho_2\dot{\eta}_2\sin\lambda_2\sin\varphi_{32}}{c^4\tau_{32}\rho_3\cos\lambda_3} - \frac{\ddot{\phi}_3}{c^3}\right] \equiv G_{\phi_3};$$
$$\dot{\eta}_3 = \frac{e_3 e_2}{m_3}\frac{\rho_2\cos\varphi_{32}\sin\lambda_3\cos\lambda_2}{c^3\tau_{32}{}^3}\frac{\left\langle \vec{\xi}^{(32)}, \dot{\vec{u}}^{(2)} \right\rangle}{c^3\rho_3} +$$
$$+\frac{e_3 e_2}{m_3 c^3}\frac{\dot{r}_3(\cos\varphi_{32}\cos\lambda_2\sin\lambda_3 - \sin\lambda_2\cos\lambda_3) + \rho_2\dot{\phi}_2\sin\varphi_{32}\sin\lambda_3\cos\lambda_2 - \rho_2\dot{\eta}_2(\cos\varphi_{23}\sin\lambda_2\sin\lambda_3 + \cos\lambda_2\cos\lambda_3}{c\tau_{32}\rho_3} - \frac{e_3^2}{m_3}\frac{\ddot{\eta}_3}{c^3} \equiv G_{\eta_3}.$$

Further on, we refer to the above system as a (BS)—basic system. It is a first-order neutral system with respect to spherical velocities.

4. Introducing Function Spaces and Some Auxiliary Propositions

It is proved that for Lipschitz continuous T-periodic trajectories, each of the functional equations,

$$\tau_{23}(t) = \frac{1}{c}\sqrt{\sum_{\alpha=1}^{3}\left[x_\alpha^{(2)}(t) - x_\alpha^{(3)}(t-\tau_{23}(t))\right]^2}, \tau_{32}(t) = \frac{1}{c}\sqrt{\sum_{\alpha=1}^{3}\left[x_\alpha^{(3)}(t) - x_\alpha^{(2)}(t-\tau_{32}(t))\right]^2}$$

has a unique T-periodic solution.

By $C_T{}^{\infty}[0,\infty)$, we denote the set of all infinitely differentiable functions, satisfying the condition $r(-T) = r(0) = 0$. We introduce the following sets, where $0 < \delta < \frac{\pi}{2}, \mu > \omega > 0$ are strictly positive constants, $r^{(m)}$ means the m-th derivative of r and $r^{(0)}(t) = r(t)$.

Then, we introduce the following function spaces:

$$M_{r_n} = \left\{ r_n \in C_T{}^{\infty}[0,\infty) : \left| r_n{}^{(m)}(t) \right| \le \omega_n{}^m R_n e^{\mu(t-T_k)};\ r_n{}^{(0)}(T_k) = 0;\ \int_{T_k}^{T_{k+1}} r_n(s)ds = 0 \right\},\ t \in [T_k, T_{k+1}];$$

$$d_{(k,m)}(r_n, \overline{r}_n) = \sup\left\{ \frac{\left|r_n{}^{(m)}(t) - \overline{r}_n{}^{(m)}(t)\right|}{\omega_n{}^m} e^{-\mu(t-T_k)} : t \in [T_k, T_{k+1}] \right\};$$

$$M_{\phi_n} = \left\{ \phi_n \in C_T{}^{\infty}[0,\infty) : \left| \phi_n{}^{(m)}(t) \right| \le \omega_n{}^m \Phi_n e^{\mu(t-T_k)},\ \int_{T_k}^{T_{k+1}} \phi_n(s)ds = T\phi_{n0} \right\},\ t \in [T_k, T_{k+1}];$$

$$d_{(k,m)}(\phi_n, \overline{\phi}_n) = \sup\left\{ \frac{\left|\phi_n{}^{(m)}(t) - \overline{\phi}_n{}^{(m)}(t)\right|}{\omega_n{}^m} e^{-\mu(t-T_k)} : t \in [T_k, T_{k+1}] \right\};$$

$$M_{\eta_n} = \left\{ \eta_n \in C_T{}^{\infty}[0,\infty) : \left| \eta_n{}^{(m)}(t) \right| \le \omega^m Y_n e^{\mu(t-T_k)};\ \eta_n{}^{(0)}(T_k) = 0;\ \int_{T_k}^{T_{k+1}} \eta_n(s)ds = 0;\ \left|\int_{T_k}^{t} \eta(s)ds\right| \le \frac{\pi}{2} - \delta \right\},\ t \in [T_k, T_{k+1}];$$

$$d_{(k,m)}(\eta_n, \overline{\eta}_n) = \sup\left\{ \frac{\left|\eta_n{}^{(m)}(t) - \overline{\eta}_n{}^{(m)}(t)\right|}{\omega_n{}^m} e^{-\mu(t-T_k)} : t \in [T_k, T_{k+1}] \right\}$$

$(k = 0, 1, 2, \ldots), \ (m = 0, 1, 2, \ldots), (n = 2, 3)$.

Here, ω_n, R_n, Φ_n, Y_n $(n = 2, 3)$ and μ are strictly positive constants, such that $\mu > \omega_n$.

We consider the product of uniform spaces, $M_{r_2} \times M_{\phi_2} \times M_{\eta_2} \times M_{r_3} \times M_{\phi_3} \times M_{\eta_3}$, endowed with the countable family of pseudo-metrics:

$$d_{(k,m)}((r_2, \phi_2, \eta_2, r_3, \phi_3, \eta_3), (\overline{r}_2, \overline{\phi}_2, \overline{\eta}_2, \overline{r}_3, \overline{\phi}_3, \overline{\eta}_3)) =$$
$$= d_{(k,m)}(r_2, \overline{r}_2) + d_{(k,m)}(\phi_2, \overline{\phi}_2) + d_{(k,m)}(\eta_2, \overline{\eta}_2) + d_{(k,m)}(r_3, \overline{r}_3) + d_{(k,m)}(\phi_3, \overline{\phi}_3) + d_{(k,m)}(\eta_3, \overline{\eta}_3)$$

where the index set A consists of all ordered pairs (k, m).

Lemma 1 ([20]]). *If* $r_n(.) \in C_T{}^{\infty}[0,\infty)$ and $\int_0^T r_n(t)dt = 0$, *then,* $\rho_n(t) = \rho_{n0} + \int_0^t r_n(s)ds$ $(n = 2, 3)$ *is a T-periodic function. In addition,*

$$\varphi_2(t+T) = \varphi_{20} + \int_0^{t+T} \phi_2(s)ds = \varphi_{20} + \int_0^t \phi_2(s)ds + \int_{T_k}^{T_{k+1}} \phi_2(s)ds = \varphi_2(t) + \phi_{20}T = \varphi_2(t) + 2\pi \Rightarrow$$
$$\Rightarrow \cos\varphi_2(t+T) = \cos(\varphi_2(t) + 2\pi) = \cos\varphi_2(t)$$

$$\varphi_3(t+T-\tau_{23}(t+T)) = \varphi_3(t+T-\tau_{23}(t)) = \varphi_{30} + \int_0^{t+T-\tau_{23}(t)} \phi_3(s)ds = \varphi_{30} + \int_0^{t-\tau_{23}(t)} \phi_3(s)ds + \int_{t-\tau_{23}(t)}^{t-\tau_{23}(t)+T} \phi_3(s)ds =$$
$$= \varphi_3(t-\tau_{23}) + \phi_{30}T = \varphi_3(t-\tau_{23}) + 2\pi \Rightarrow$$
$$\cos\varphi_3(t+T-\tau_{23}(t+T)) = \cos(\varphi_3(t-\tau_{23}) + 2\pi) = \cos\varphi_3(t-\tau_{23}).$$

It follows that the right-hand sides of the equations of (BS) are T-periodic functions.

Lemma 2. *The distances* $\rho_n(t)$, $(n = 2, 3)$ *are bounded functions.*

Proof. For sufficiently large $\mu > 0$ and $\mu T = const. > 0$, we obtain

$$\rho_n(t) \ge \rho_{n0} - \int_0^t |r_\alpha(s)|ds \ge \rho_{n0} - R_n \int_0^t e^{\mu s}ds = \rho_{n0} - R_n \frac{e^{\mu t} - 1}{\mu} \ge \rho_{n0} - \frac{R_n(e^{\mu T} - 1)}{\mu} \ge \rho_{n0} - \frac{R_n e^{\mu T}}{\mu} \equiv \widehat{\rho}_n > 0, \quad (n = 2, 3).$$

In a similar way, we infer

$$\rho_n(t) \le \rho_{n0} + \int_0^t |r_n(s)|ds \le \rho_{n0} + R_n \int_0^t e^{\mu s} ds \le \rho_{n0} + R_n \frac{e^{\mu T} - 1}{\mu} \equiv \breve{\rho}_n > 0\ , \quad (n = 2, 3).$$

Finally, $\widehat{\rho}_n < \breve{\rho}_n$ since $e^{\mu T} - 1 > 0$.
Lemma 2 is thus proved. □

Lemma 3. *If* $\int_0^T \phi_n(t)dt = \phi_{n0}T\ (\phi_{n0} > 0), (n = 2, 3)$, *then* $\lim\limits_{t\to\infty} \varphi_n(t) = \infty$.

Proof. Let us set $s + T = p$. Then, we obtain $\int_0^T \phi_n(s)ds = \int_T^{2T} \phi_n(p)dp = \ldots = \int_{T_k}^{T_{k+1}} \phi_n(p)dp$. Therefore, $\lim\limits_{t\to\infty} \varphi_n(t) = \lim\limits_{t\to\infty}\left(\varphi_{n0} + \int_0^\infty \phi_n(s)ds\right) = \varphi_{n0} + \sum_{k=0}^{\infty} \int_{T_k}^{T_{k+1}} \phi_n(s)ds = \varphi_{n0} + T\phi_{n0} \sum_{k=1}^{\infty} 1 = \infty$.
Lemma 3 is thus proved. □

Lemma 4. *The sets* M_{r_n}, M_{ϕ_n}, M_{η_n} $(n = 2, 3)$ *are closed.*

For the proof, cf. [19].
The following assertions give a qualitative estimate of the electron orbits.

Lemma 5. *The following inequality is valid:*
$A = |\cos\varphi_2 \cos\varphi_3 \cos\lambda_2 \cos\lambda_3 + \sin\varphi_2 \sin\varphi_3 \cos\lambda_2 \cos\lambda_3 + \sin\lambda_2 \sin\lambda_3| \le 1.$

Proof.

$$\begin{aligned} A &= |\cos\lambda_2 \cos\lambda_3 \cos(\varphi_2 - \varphi_3) + \sin\lambda_2 \sin\lambda_3| = \\ &= \left|\tfrac{\cos(\lambda_2-\lambda_3)+\cos(\lambda_2+\lambda_3)}{2}\cos(\varphi_2-\varphi_3) + \tfrac{\cos(\lambda_2-\lambda_3)-\cos(\lambda_2+\lambda_3)}{2}\right| = \\ &= \left|\tfrac{\cos(\lambda_2-\lambda_3)[1+\cos(\varphi_2-\varphi_3)]-\cos(\lambda_2+\lambda_3)[1-\cos(\varphi_2-\varphi_3)]}{2}\right| = \\ &= \tfrac{1}{2}\left|2\cos^2\tfrac{\varphi_2-\varphi_3}{2}\cos(\lambda_2-\lambda_3) - 2\sin^2\tfrac{\varphi_2-\varphi_3}{2}\cos(\lambda_2+\lambda_3)\right| \le \cos^2\tfrac{\varphi_2-\varphi_3}{2} + \sin^2\tfrac{\varphi_2-\varphi_3}{2} = 1. \end{aligned}$$

□

Lemma 6. *The following inequalities are valid:*
$|\rho_{20} - \rho_{30}| - (R_2 + R_3)\frac{e^{\mu T}-1}{\mu} \le \rho_{23}(t), \rho_{32}(t) \le \rho_{20} + \rho_{30} + (R_2 + R_3)\frac{e^{\mu T}-1}{\mu}$.

Proof. Since $\rho_2(t) = \rho_{20} + \int_0^t r_2(s)ds$, $\rho_3(t) = \rho_{30} + \int_0^t r_3(s)ds$, then, in view of Lemma 2,

$$\begin{aligned} \rho_{23}(t) &= \sqrt{(\rho_2\cos\varphi_2\cos\lambda_2 - \rho_3\cos\varphi_3\cos\lambda_3)^2 + (\rho_2\sin\varphi_2\cos\lambda_2 - \rho_3\sin\varphi_3\cos\lambda_3)^2 + (\rho_2\sin\lambda_2 - \rho_3\sin\lambda_3)^2} \le \\ &\le \sqrt{\rho_2{}^2 + \rho_3{}^2 + 2\rho_2\rho_3 A} \le \sqrt{\rho_2{}^2 + \rho_3{}^2 + 2\rho_2\rho_3} = \rho_2(t) + \rho_3(t) \le \rho_{20} + \rho_{30} + (R_2 + R_3)\tfrac{e^{\mu T}-1}{\mu} \end{aligned}$$

and

$$\begin{aligned} \rho_{23}(t) &= \sqrt{(\rho_2\cos\varphi_2\cos\lambda_2 - \rho_3\cos\varphi_3\cos\lambda_3)^2 + (\rho_2\sin\varphi_2\cos\lambda_2 - \rho_3\sin\varphi_3\cos\lambda_3)^2 + (\rho_2\sin\lambda_2 - \rho_3\sin\lambda_3)^2} \ge \\ &\ge \sqrt{\rho_2{}^2 + \rho_3{}^2 - 2\rho_2\rho_3 A} \ge \sqrt{\rho_2{}^2 + \rho_3{}^2 - 2\rho_2\rho_3} = |\rho_2 - \rho_3| \ge |\rho_{20} - \rho_{30}| - (R_2 + R_3)\tfrac{e^{\mu T}-1}{\mu} \end{aligned}$$

which completes the proof. □

Lemma 7. *The following inequalities are valid:*

$$\tau_{23} \geq \frac{|\rho_2(t)-\rho_3(t-\tau_{23})|}{c} = \frac{\left|\rho_{02}-\rho_{03}+\int\limits_0^t r_2(s)ds-\int\limits_0^t r_3(s)ds-\int\limits_t^{t-\tau_{23}} r_3(s)ds\right|}{c} \geq \frac{|\rho_{02}-\rho_{03}|-(R_2+R_3)\int\limits_0^t e^{\mu s}ds-\left|\int\limits_t^{t-\tau_{23}} r_3(s)ds\right|}{c} \geq$$

$$\geq \frac{|\rho_{02}-\rho_{03}|-(R_2+R_3)\frac{e^{\mu T}-1}{\mu}-R_3\left|\frac{e^{\mu(t-\tau_{23})}-e^{\mu t}}{\mu}\right|}{c} \geq \frac{|\rho_{02}-\rho_{03}|-(R_2+R_3)\frac{e^{\mu T}-1}{\mu}-R_3e^{\mu t}\left|\frac{e^{-\mu\tau_{23}}-1}{\mu}\right|}{c} \geq$$

$$\geq \frac{|\rho_{02}-\rho_{03}|-R_2\frac{e^{\mu T}-1}{\mu}-R_3\frac{e^{\mu T}-1}{\mu}-R_3\frac{e^{\mu T}}{\mu}}{c} = \frac{|\rho_{02}-\rho_{03}|-R_2\frac{e^{\mu T}-1}{\mu}-R_3\frac{e^{\mu T}}{\mu}}{c} \geq \frac{|\rho_{02}-\rho_{03}|-(R_2+R_3)\frac{e^{\mu T}}{\mu}}{c} \equiv \frac{\Delta}{c} > 0$$

and

$$\tau_{32} \geq \frac{|\rho_3(t)-\rho_2(t-\tau_{32})|}{c} = \frac{\left|\rho_{03}-\rho_{02}+\int\limits_0^t r_3(s)ds-\int\limits_0^t r_2(s)ds-\int\limits_t^{t-\tau_{32}} r_2(s)ds\right|}{c} \geq \frac{|\rho_{03}-\rho_{02}|-(R_2+R_3)\int\limits_0^t e^{\mu s}ds-\left|\int\limits_t^{t-\tau_{32}} r_2(s)ds\right|}{c} \geq$$

$$\geq \frac{|\rho_{03}-\rho_{02}|-(R_3+R_2)\frac{e^{\mu T}-1}{\mu}-R_2\left|\frac{e^{\mu(t-\tau_{32})}-e^{\mu t}}{\mu}\right|}{c} \geq \frac{|\rho_{03}-\rho_{02}|-(R_3+R_2)\frac{e^{\mu T}-1}{\mu}-R_2e^{\mu t}\left|\frac{e^{-\mu\tau_{32}}-1}{\mu}\right|}{c} \geq$$

$$\geq \frac{|\rho_{03}-\rho_{02}|-R_3\frac{e^{\mu T}-1}{\mu}-R_2\frac{e^{\mu T}-1}{\mu}-R_2\frac{e^{\mu T}}{\mu}}{c} = \frac{|\rho_{03}-\rho_{02}|-R_3\frac{e^{\mu T}-1}{\mu}-R_2\frac{e^{\mu T}}{\mu}}{c} \geq \frac{|\rho_{02}-\rho_{03}|-(R_2+R_3)\frac{e^{\mu T}}{\mu}}{c} \equiv \frac{\Delta}{c} > 0$$

for sufficiently large $\mu > 0$ *and* $\mu T = const. > 0$. *The above inequalities imply that* $|\rho_2(t) - \rho_3(t - \tau_{23})|$, $|\rho_3(t) - \rho_2(t - \tau_{32})|$ *are strictly positive, which completes the proof.*

Remark 3. *Clearly,* $\frac{1}{\tau_{23}} \leq \frac{c}{\Delta}$, $\frac{1}{\tau_{32}} \leq \frac{c}{\Delta}$.

5. Introducing Operators and the Main Lemma

Prior to the formulation and proving of the Main Lemma, we give the compatibility condition (CC), which implies the continuity of solution for neutral equations: the initial functions satisfy the following relations:

$$\dot{r}_{20}(0) = \frac{e_2e_1}{m_2}\frac{1}{c\rho_{20}{}^2(0)} -$$

$$-\frac{e_2e_3}{m_2}\frac{\rho_{20}(0)-\rho_{30}(-\tau_{23}(0))(\cos\varphi_{23}\cos\lambda_{20}(0)\cos\lambda_{30}(-\tau_{23}(0))+\sin\lambda_{20}(0)\sin\lambda_{30}(-\tau_{23}(0)))}{c^3}\frac{\left\langle \vec{\xi}^{(23)}, \dot{\vec{u}}^{(3)}\right\rangle}{c^3\tau_{23}{}^3}$$

$$-\frac{e_2e_3}{m_2}\frac{\dot{r}_{30}(-\tau_{23}(0))(\cos\varphi_{23}\cos\lambda_{20}(0)\cos\lambda_{30}(-\tau_{23}(0))+\sin\lambda_{20}(0)\sin\lambda_{30}(-\tau_{23}(0)))}{c^4\tau_{23}}$$

$$-\frac{e_2e_3}{m_2}\frac{\rho_{30}(-\tau_{23}(0))\dot{\phi}_{30}(-\tau_{23}(0))\sin\varphi_{23}\cos\lambda_{20}(0)\cos\lambda_{30}(-\tau_{23}(0))}{c^4\tau_{23}}$$

$$-\frac{e_2e_3}{m_2}\frac{\rho_{30}(-\tau_{23}(0))\dot{\eta}_{30}(-\tau_{23}(0))(\sin\lambda_{20}(0)\cos\lambda_{30}(-\tau_{23}(0))-\cos\varphi_{23}\sin\lambda_{30}(-\tau_{23}(0))\cos\lambda_{20}(0))}{c^4\tau_{23}} - \frac{e_2^2}{m_2}\frac{\ddot{r}_{20}(0)}{c^3} \equiv G_{r_2}(0);$$

where $\varphi_{23} = \varphi_2(0) - \varphi_3(-\tau_{23}(0))$ and so on.

Define an operator B acting on $M_{r_2} \times M_{\phi_2} \times M_{\eta_2} \times M_{r_3} \times M_{\phi_3} \times M_{\eta_3}$ by the formulas

$$B_{r_2}^{(k)}(r_2,\phi_2,\eta_2,r_3,\phi_3,\eta_3)(t) := \begin{cases} \int\limits_{T_k}^{t} G_{r_2}(s)ds - \left(\frac{t-T_k}{T}-\frac{1}{2}\right)\int\limits_{T_k}^{T_{k+1}} G_{r_2}(s)ds - \frac{1}{T}\int\limits_{T_k}^{T_{k+1}}\int\limits_{T_k}^{p} G_{r_2}(s)dsdp,\ t\in[T_k,T_{k+1}] \\ r_3(t)=r_{30}(t),\phi_3(t)=\phi_{30}(t),\eta_3(t)=\eta_{30}(t),\ t\in[-T,0] \end{cases};$$

$$B_{\phi_2}^{(k)}(r_2,\phi_2,\eta_2,r_3,\phi_3,\eta_3)(t) := \begin{cases} \phi_{20} + \int\limits_{T_k}^{t} G_{\phi_2}(s)ds - \left(\frac{t-T_k}{T}-\frac{1}{2}\right)\int\limits_{T_k}^{T_{k+1}} G_{\phi_2}(s)ds - \frac{1}{T}\int\limits_{T_k}^{T_{k+1}}\int\limits_{T_k}^{p} G_{\phi_2}(s)dsdp,\ t\in[T_k,T_{k+1}] \\ r_3(t)=r_{30}(t),\phi_3(t)=\phi_{30}(t),\eta_3(t)=\eta_{30}(t),\ t\in[-T,0] \end{cases};$$

$$B_{\eta_2}^{(k)}(r_2,\phi_2,\eta_2,r_3,\phi_3,\eta_3)(t) := \begin{cases} \int\limits_{T_k}^{t} G_{r_3}(s)ds - \left(\frac{t-T_k}{T}-\frac{1}{2}\right)\int\limits_{T_k}^{T_{k+1}} G_{r_3}(s)ds - \frac{1}{T}\int\limits_{T_k}^{T_{k+1}}\int\limits_{T_k}^{p} G_{r_3}(s)dsdp,\ t\in[T_k,T_{k+1}] \\ r_3(t)=r_{30}(t),\phi_3(t)=\phi_{30}(t),\eta_3(t)=\eta_{30}(t),\ t\in[-T,0] \end{cases};$$

$$B_{r_3}^{(k)}(r_2,\phi_2,\eta_2,r_3,\phi_3,\eta_3)(t) := \begin{cases} \int\limits_{T_k}^{t} G_{r_3}(s)ds - \left(\frac{t-T_k}{T}-\frac{1}{2}\right)\int\limits_{T_k}^{T_{k+1}} G_{r_3}(s)ds - \frac{1}{T}\int\limits_{T_k}^{T_{k+1}}\int\limits_{T_k}^{p} G_{r_3}(s)dsdp,\ t\in[T_k,T_{k+1}] \\ r_2(t)=r_{20}(t),\phi_2(t)=\phi_{20}(t),\eta_2(t)=\eta_{20}(t),\ t\in[-T,0] \end{cases};$$

$$B_{\phi_3}^{(k)}(r_2,\phi_2,\eta_2,r_3,\phi_3,\eta_3)(t) := \begin{cases} \phi_{30} + \int\limits_{T_k}^{t} G_{\phi_3}(s)ds - \left(\frac{t-T_k}{T}-\frac{1}{2}\right)\int\limits_{T_k}^{T_{k+1}} G_{\phi_3}(s)ds - \frac{1}{T}\int\limits_{T_k}^{T_{k+1}}\int\limits_{T_k}^{p} G_{\phi_3}(s)dsdp, & t\in[T_k,T_{k+1}]; \\ r_2(t)=r_{20}(t),\phi_2(t)=\phi_{20}(t),\eta_2(t)=\eta_{20}(t),\ t\in[-T,0] \end{cases}$$

$$B_{\eta_3}^{(k)}(r_2,\phi_2,\eta_2,r_3,\phi_3,\eta_3)(t) := \begin{cases} \int\limits_{T_k}^{t} G_{\eta_3}(s)ds - \left(\frac{t-T_k}{T}-\frac{1}{2}\right)\int\limits_{T_k}^{T_{k+1}} G_{\eta_3}(s)ds - \frac{1}{T}\int\limits_{T_k}^{T_{k+1}}\int\limits_{T_k}^{p} G_{\eta_3}(s)dsdp,\ t\in[T_k,T_{k+1}] \\ r_2(t)=r_{20}(t),\phi_2(t)=\phi_{20}(t),\eta_2(t)=\eta_{20}(t),\ t\in[-T,0] \end{cases}$$

where $t\in[T_k,T_{k+1}]$, $T_k = kT$ $(k=0,1,2,\dots)$, $G_{\phi_2} = G_{\phi_2}(r_2,\phi_2,\eta_2,r_3,\phi_3,\eta_3)$, $G_{\eta_2} = G_{\eta_2}(r_2,\phi_2,\eta_2,r_3,\phi_3,\eta_3)$, $G_{r_3} = G_{r_3}(r_2,\phi_2,\eta_2,r_3,\phi_3,\eta_3)$, $G_{\phi_3} = G_{\phi_3}(r_2,\phi_2,\eta_2,r_3,\phi_3,\eta_3)$, $G_{\eta_3} = G_{\eta_3}(r_2,\phi_2,\eta_2,r_3,\phi_3,\eta_3)$ and the initial functions $r_{n0}(t),\phi_{n0}(t),\eta_{n0}(t),(n=2,3)$ are solutions of the system of initial equations.

Remark 4. *We notice that in view of the periodicity of $G_{r_2}(s)$, the operator $B_{r_2}(.)(t)$ is T-periodic on $[0,\infty)$ (cf. [20]). The conformity condition (CC) $\dot r_{20}(0) = G_{r_2}(0)$ implies*

$$\dot r_2(t) = G_{r_2}(t) - \frac{1}{T}\int\limits_0^T G_{r_2}(s)ds \Leftrightarrow \dot r_{20}(0) = G_{r_2}(0) - \frac{1}{T}\int\limits_0^T G_{r_2}(s)ds \Leftrightarrow \int\limits_0^T G_{r_2}(s)ds = 0$$

and

$$r_2(t) = B_{r_2}^{(0)}(.)(t) \Rightarrow r_2(0) = B_{r_2}^{(0)}(.)(0) \Leftrightarrow 0 = \frac{1}{2}\int\limits_0^T G_{r_2}(s)ds - \frac{1}{T}\int\limits_0^T\int\limits_0^p G_{r_2}(s)dsdp \Rightarrow \int\limits_0^T\int\limits_0^p G_{r_2}(s)dsdp = 0$$

$$B_{r_2}^{(0)}(.)(0) = \frac{1}{2}\int\limits_0^T G_{r_2}(s)ds - \frac{1}{T}\int\limits_0^T\int\limits_0^p G_{r_2}(s)dsdp = 0;\ B_{r_2}^{(0)}(.)(T) = \frac{1}{2}\int\limits_0^T G_{r_2}(s)ds - \frac{1}{T}\int\limits_0^T\int\limits_0^p G_{r_2}(s)dsdp = 0.$$

But $G_{r_2}(s)$ is a T-periodic function. Therefore, one can extend the result for every $k=1,2,\dots$, that is, $\int\limits_{T_k}^{T_{k+1}} G_{r_2}(s)ds = \int\limits_{T_k}^{T_{k+1}}\int\limits_{T_k}^{p} G_{r_2}(s)dsdp = 0$.

We need the following inequalities:

$$\left\langle \overset{\rightarrow(23)}{\xi}, \overset{\dot{\rightarrow}(3)}{u} \right\rangle \leq \sqrt{\left\langle \overset{\rightarrow(23)}{\xi}, \overset{\rightarrow(23)}{\xi} \right\rangle}\sqrt{\left\langle \overset{\dot{\rightarrow}(3)}{u}, \overset{\dot{\rightarrow}(3)}{u} \right\rangle} \leq c\tau_{23}\sqrt{\ddot{\rho}_3{}^2 + \rho_3{}^2\ddot{\varphi}_3{}^2\cos^2\lambda_3 + \rho_3{}^2\ddot{\lambda}_3{}^2} =$$

$$= c\tau_{23}\sqrt{\dot{r}_3{}^2 + \rho_3{}^2\dot{\phi}_3{}^2\cos^2\lambda_3 + \rho_3{}^2\dot{\eta}_3{}^2} \leq c\tau_{23}\omega_3 e^{\mu(t-T_k)}\sqrt{R_3{}^2 + \rho_3{}^2 R_3{}^2 + \rho_3{}^2 Y_3{}^2} \leq \bar{c}c\tau_{23}\omega_3 e^{\mu(t-T_k)}$$

$$\left\langle \overset{\rightarrow(32)}{\xi}, \overset{\dot{\rightarrow}(2)}{u} \right\rangle \leq \sqrt{\left\langle \overset{\rightarrow(32)}{\xi}, \overset{\rightarrow(32)}{\xi} \right\rangle}\sqrt{\left\langle \overset{\dot{\rightarrow}(2)}{u}, \overset{\dot{\rightarrow}(2)}{u} \right\rangle} \leq c\tau_{32}\sqrt{\ddot{\rho}_2{}^2 + \rho_2{}^2\ddot{\varphi}_2{}^2\cos^2\lambda_2 + \rho_2{}^2\ddot{\lambda}_2{}^2} =$$

$$= c\tau_{23}\sqrt{\dot{r}_3{}^2 + \rho_3{}^2\dot{\phi}_3{}^2\cos^2\lambda_3 + \rho_3{}^2\dot{\eta}_3{}^2} \leq c\tau_{32}\omega_2 e^{\mu(t-T_k)}\sqrt{R_2{}^2 + \rho_2{}^2 R_2{}^2 + \rho_2{}^2 Y_2{}^2} \leq \bar{c}c\tau_{32}\omega_2 e^{\mu(t-T_k)}$$

In addition, the condition (C) implies $\left\langle \overset{\rightarrow(n)}{u}, \overset{\rightarrow(n)}{u} \right\rangle = r_n{}^2 + \rho_n{}^2\phi_n{}^2\cos^2\lambda_n + \rho_n{}^2\eta_n{}^2 \leq R_n{}^2 + \rho_n{}^2\Phi_n{}^2 + \rho_n{}^2 Y_n{}^2 \leq \bar{c}^2 < c^2, (n = 2,3).$

Lemma 8. *The following inequalities are valid:*

$$\left|\int_{T_k}^{t} G_{r_2}(s)ds\right| \leq \left|\left(\int_{T_k}^{t} \frac{|e_2e_1|}{m_2 c}\frac{1}{\widehat{\rho}_2{}^2} + \frac{|e_2e_3|}{m_2}\frac{\bar{c}c\tau_{23}\omega_3 e^{\mu(t-T_k)}}{c^6}\frac{\rho_2+\rho_3}{\tau_{23}{}^3} + \frac{|e_2e_3|}{m_2}\frac{|\dot{r}_3| + |\dot{\phi}_3\rho_3| + |\dot{\eta}_3\rho_3|}{c^4\tau_{23}} + \frac{e_2^2}{m_2 c^3}|\ddot{r}_2|\right)ds\right| \leq$$

$$\leq \frac{|e_2e_1|}{m_2}\left|\int_{T_k}^{t}\left(\frac{1}{c\widehat{\rho}_2{}^2} + \frac{\bar{c}\omega_3(\breve{\rho}_2+\breve{\rho}_3+3\Delta)}{c^3\Delta^2} + \frac{\omega_2{}^2R_2}{c^3}\right)e^{\mu(s-T_k)}ds\right| \leq \frac{|e_2e_1|}{m_2}\left(\frac{1}{c\widehat{\rho}_2{}^2} + \frac{\bar{c}\omega_3(\breve{\rho}_2+\breve{\rho}_3+3\Delta)}{c^3\Delta^2} + \frac{\omega_2{}^2R_2}{c^3}\right)\frac{e^{\mu(t-T_k)}}{\mu};$$

$$\left|\int_{T_k}^{T_{k+1}} G_{r_2}(s)ds\right| \leq \frac{|e_2e_1|}{m_2}\left(\frac{1}{c\widehat{\rho}_2{}^2} + \frac{\bar{c}\omega_3(\breve{\rho}_2+\breve{\rho}_3+3\Delta)}{c^2\Delta^2} + \frac{\omega_2{}^2R_2}{c^3}\right)\frac{e^{\mu T}-1}{\mu};$$

$$\left|\int_{T_k}^{t} G_{\phi_2}ds\right| \leq \frac{|e_2e_3|}{m_2}\int_{T_k}^{t}\left(\frac{\rho_3\bar{c}\omega_3}{c^5\tau_{23}{}^2\rho_2\cos\lambda_2} + \frac{|\dot{r}_3| + |\rho_3\dot{\phi}_3| + |\rho_3\dot{\eta}_3|}{c^4\tau_{23}\rho_2\cos\lambda_2} + \frac{|\ddot{\phi}_2|}{c^3}\right)e^{\mu(s-T_k)}ds \leq \frac{|e_2e_3|}{m_2}\left(\frac{\bar{c}\omega_3(\breve{\rho}_3+3\Delta)}{\widehat{\rho}_2 c^3\Delta^2\cos(\pi/2-\delta)} + \frac{\omega_2{}^2\Phi_2}{c^3}\right)\frac{e^{\mu(t-T_k)}}{\mu};$$

$$\left|\int_{T_k}^{T_{k+1}} G_{\phi_2}ds\right| \leq \frac{|e_2e_3|}{m_2}\left(\frac{\bar{c}\omega_3(\breve{\rho}_3+3\Delta)}{\widehat{\rho}_2 c^3\Delta^2\cos(\pi/2-\delta)} + \frac{\omega_2{}^2\Phi_2}{c^3}\right)\frac{e^{\mu T}-1}{\mu};$$

$$\left|\int_{T_k}^{t} G_{\eta_2}ds\right| \leq \left|\frac{e_2e_3}{m_2}\right|\int_{T_k}^{t}\left(\frac{\rho_3\bar{c}\omega_3 e^{\mu(s-T_k)}}{c^5\tau_{23}{}^2\rho_2} + \frac{|\dot{r}_3| + |\rho_3\dot{\phi}_3| + |\rho_3\dot{\eta}_3|}{c^4\tau_{23}\rho_2} + \frac{\omega_2{}^2Y_2}{c^3}\right)e^{\mu(s-T_k)}ds \leq \frac{|e_2e_3|}{m_2}\left(\frac{\bar{c}\omega_3}{c^3}\frac{\breve{\rho}_3 e^{\mu T}+3\Delta}{\widehat{\rho}_2\Delta^2} + \frac{\omega_2{}^2Y_2}{c^3}\right)\frac{e^{\mu(t-T_k)}}{\mu};$$

$$\left|\int_{T_k}^{T_{k+1}} G_{\eta_2}ds\right| \leq \frac{|e_2e_3|}{m_2}\left(\frac{\bar{c}\omega_3}{c^3}\frac{\breve{\rho}_3 e^{\mu T}+3\Delta}{\widehat{\rho}_2\Delta^2} + \frac{\omega_2{}^2Y_2}{c^3}\right)\frac{e^{\mu T}-1}{\mu};$$

$$\left|\int_{T_k}^{t} G_{r_3}(s)ds\right| \leq \frac{|e_2e_1|}{m_3}\left(\frac{1}{c\widehat{\rho}_3{}^2} + \frac{\bar{c}\omega_2(\breve{\rho}_2+\breve{\rho}_3+3\Delta)}{c^2\Delta^2} + \frac{\omega_3{}^2R_3}{c^3}\right)\frac{e^{\mu(t-T_k)}}{\mu};$$

$$\left|\int_{T_k}^{T_{k+1}} G_{r_3}(s)ds\right| \leq \frac{|e_2e_1|}{m_3}\left(\frac{1}{c\widehat{\rho}_3{}^2} + \frac{\bar{c}\omega_2(\breve{\rho}_2+\breve{\rho}_3+3\Delta)}{c^2\Delta^2} + \frac{\omega_3{}^2R_3}{c^3}\right)\frac{e^{\mu T}-1}{\mu};$$

$$\left|\int_{T_k}^{t} G_{\phi_3}ds\right| \leq \frac{|e_2e_3|}{m_3}\left(\frac{\bar{c}\omega_2(\breve{\rho}_2+3\Delta)}{\widehat{\rho}_3 c^3\Delta^2\cos(\pi/2-\delta)} + \frac{\omega_3{}^2\Phi_3}{c^3}\right)\frac{e^{\mu(t-T_k)}}{\mu};$$

$$\left|\int_{T_k}^{T_{k+1}} G_{\phi_3}ds\right| \leq \frac{|e_2e_3|}{m_3}\left(\frac{\bar{c}\omega_2(\breve{\rho}_2+3\Delta)}{\widehat{\rho}_3 c^3\Delta^2\cos(\pi/2-\delta)} + \frac{\omega_3{}^2\Phi_3}{c^3}\right)\frac{e^{\mu T}-1}{\mu};\left|\int_{T_k}^{t} G_{\eta_3}ds\right| \leq \frac{|e_2e_3|}{m_3}\left(\frac{\bar{c}\omega_2}{c^3}\frac{\breve{\rho}_2 e^{\mu T}+3\Delta}{\widehat{\rho}_3\Delta^2} + \frac{\omega_3{}^2Y_3}{c^3}\right)\frac{e^{\mu(t-T_k)}}{\mu};$$

$$\left|\int_{T_k}^{T_{k+1}} G_{\eta_3}ds\right| \leq \frac{|e_2e_3|}{m_3}\left(\frac{\bar{c}\omega_2}{c^3}\frac{\breve{\rho}_2 e^{\mu T}+3\Delta}{\widehat{\rho}_3\Delta^2} + \frac{\omega_3{}^2Y_3}{c^3}\right)\frac{e^{\mu T}-1}{\mu}.$$

Lemma 9 (Main Lemma). *Let the compatibility condition (CC) be satisfied. The periodic problem for the system (BS) has a solution* $(r_2, \phi_2, \eta_2, r_3, \phi_3, \eta_3) \in M_{r_2} \times M_{\phi_2} \times M_{\eta_2} \times M_{r_3} \times M_{\phi_3} \times M_{\eta_3}$ *if the operator B has a fixed point belonging to* $M_{r_2} \times M_{\phi_2} \times M_{\eta_2} \times M_{r_3} \times M_{\phi_3} \times M_{\eta_3}$.

Proof. Let the system (BS) have a solution belonging to $M_{r_2} \times M_{\phi_2} \times M_{\eta_2} \times M_{r_3} \times M_{\phi_3} \times M_{\eta_3}$.
□

We show that $B = \left(B_{r_2}, B_{\phi_2}, B_{\eta_2}, B_{r_3}, B_{\phi_3}, B_{\eta_3}\right)$ possesses a fixed point belonging to $M_{r_2} \times M_{\phi_2} \times M_{\eta_2} \times M_{r_3} \times M_{\phi_3} \times M_{\eta_3}$. Indeed, integrating the first equation of (BS), we obtain $r_2(t) = \int\limits_{T_k}^{t} G_{r_2}(s)ds$, and therefore, in view of Remark 4, one obtains

$$r_2(t) = \int\limits_{T_k}^{t} G_{r_2}(s)ds - \left(\frac{t-T_k}{T} - \frac{1}{2}\right)\int\limits_{T_k}^{T_{k+1}} G_{r_2}(s)ds - \frac{1}{T}\int\limits_{T_k}^{T_{k+1}}\int\limits_{T_k}^{p} G_{r_2}(s)dsdp\,,$$

that is, $r_2 = B_{r_2}{}^{(k)}(r_2, \phi_2, \eta_2, r_3, \phi_3, \eta_3)$.

For the second component ϕ_2, we have $\int\limits_{T_k}^{T_{k+1}} G_{\phi_2}(s)ds = 0$. Therefore,

$$\phi_2(T_{k+1}) = \phi_2(T_k) + \int\limits_{T_k}^{T_{k+1}} G_{\phi_2}(s)ds \Leftrightarrow \phi_{02} = \phi_{02} + \int\limits_{T_k}^{T_{k+1}} G_{\phi_2}(s)ds \Rightarrow \int\limits_{T_k}^{T_{k+1}} G_{\phi_2}(s)ds = 0.$$

Therefore, $\phi_2(t) = \phi_{02} + \int\limits_{T_k}^{t} G_{\phi_2}(s)ds - \left(\frac{t-T_k}{T} - \frac{1}{2}\right)\int\limits_{T_k}^{T_{k+1}} G_{\phi_2}(s)ds$. In the same way,

$$\int\limits_{T_k}^{T_{k+1}}\int\limits_{T_k}^{p} G_{\phi_2}(s)dsdp = \int\limits_{T_k}^{T_{k+1}}\int\limits_{s}^{T_{k+1}} G_{\phi_2}(s)dpds = \int\limits_{T_k}^{T_{k+1}} (T_{k+1}-s)G_{\phi_2}(s)ds = T_{k+1}\int\limits_{T_k}^{T_{k+1}} G_{\phi_2}(s)ds - \int\limits_{T_k}^{T_{k+1}} sG_{\phi_2}(s)ds = -\int\limits_{T_k}^{T_{k+1}} sG_{\phi_2}(s)ds =$$

$$= -\int\limits_{T_k}^{T_{k+1}} s\frac{d\phi_2(s)}{ds}ds = -\int\limits_{T_k}^{T_{k+1}} sd\phi_2(s) = -T_{k+1}\phi_2(T_{k+1}) + T_k\,\phi_2(T_k) + \int\limits_{T_k}^{T_{k+1}} \phi_2(s)ds = -T\phi_{02} + T\,\phi_{02}(T_k) + \int\limits_{T_k}^{T_{k+1}} \phi_2(s)ds = -T\phi_{02} + T\phi_{02} = 0.$$

Therefore,

$$\phi_2(t) = \phi_{02} + \int\limits_{T_k}^{t} G_{\phi_2}(s)ds - \left(\frac{t-T_k}{T} - \frac{1}{2}\right)\int\limits_{T_k}^{T_{k+1}} G_{\phi_2}(s)ds - \frac{1}{T}\int\limits_{T_k}^{T_{k+1}}\int\limits_{T_k}^{p} G_{\phi_2}(s)dsdp$$

which means $\phi_2 = B_{\phi_2}(r_2, \phi_2, \eta_2, r_3, \phi_3, \eta_3)$.

Repeating the same reasoning, we obtain

$$\begin{aligned}&\eta_2 = B_{\eta_2}(r_2, \phi_2, \eta_2, r_3, \phi_3, \eta_3),\ \ r_3 = B_{r_3}(r_2, \phi_2, \eta_2, r_3, \phi_3, \eta_3),\\ &\phi_3 = B_{\phi_3}(r_2, \phi_2, \eta_2, r_3, \phi_3, \eta_3),\ \ \eta_3 = B_{\eta_3}(r_2, \phi_2, \eta_2, r_3, \phi_3, \eta_3)).\end{aligned}$$

Conversely, let $(r_2, \phi_2, \eta_2, r_3, \phi_3, \eta_3) \in M_{r_2} \times M_{\phi_2} \times M_{\eta_2} \times M_{r_3} \times M_{\phi_3} \times M_{\eta_3}$ be a fixed point of B. Then,

$$r_2(t) = \int\limits_{T_k}^{t} G_{r_2}(s)ds - \left(\frac{t-T_k}{T} - \frac{1}{2}\right)\int\limits_{T_k}^{T_{k+1}} G_{r_2}(s)ds - \frac{1}{T}\int\limits_{T_k}^{T_{k+1}}\int\limits_{T_k}^{s} G_{r_2}(\theta)d\theta ds\,.$$

gives for $t = T_k$

$$r_2(T_k) = \int\limits_{T_k}^{T_k} G_{r_2}(s)ds - \left(\frac{T_k-T_k}{T} - \frac{1}{2}\right)\int\limits_{T_k}^{T_{k+1}} G_{r_2}(s)ds - \frac{1}{T}\int\limits_{T_k}^{T_{k+1}}\int\limits_{T_k}^{s} G_{r_2}(\theta)d\theta ds \Rightarrow 0 = \frac{1}{2}\int\limits_{T_k}^{T_{k+1}} G_{r_2}(s)ds - \frac{1}{T}\int\limits_{T_k}^{T_{k+1}}\int\limits_{T_k}^{s} G_{r_2}(\theta)d\theta ds.$$

But Remark 4 implies $\int\limits_{T_k}^{T_{k+1}} G_{r_2}(s)ds = 0 \Rightarrow \frac{1}{T}\int\limits_{T_k}^{T_{k+1}}\int\limits_{T_k}^{s} G_{r_2}(\theta)d\theta ds = 0$.

Consequently, $r_2(t) = \int\limits_{T_k}^{t} G_{r_2}(s)ds \Rightarrow \dot{r}_2(t) = G_{r_2}(t)$.

For the remaining components of B, we proceed in the same way.

Lemma 9 is thus proved.

6. The Main Result

The main result of the paper is as follows:

Theorem 1 (Main Theorem). *Let the compatibility condition (CC) be satisfied, and initial data be sufficiently small (Appendix C). If the following inequalities are satisfied*:

(1)
$$\frac{|e_2e_1|}{m_2}\left(\frac{1}{c\overset{\frown}{\rho}_2{}^2}+\frac{\bar{c}\omega_3(\breve{\rho}_2+\breve{\rho}_3+3|\Delta|)}{c^3\Delta^2}+\frac{\omega_2{}^2R_2}{c^3}\right)\frac{e^{\mu T}}{\mu}\le R_2\ (\mu>\omega_2,\,\omega_3);$$
$$\phi_{20}+\frac{|e_2e_1|}{m_2}\left(\frac{\bar{c}\omega_3(\breve{\rho}_3+3|\Delta|)}{\overset{\frown}{\rho}_2c^3\Delta^2\cos(\pi/2-\delta)}+\frac{\omega_2{}^2\Phi_2}{c^3}\right)\frac{e^{\mu T}}{\mu}\le\Phi_2\ (\phi_{20}>0);$$
$$\frac{|e_2e_1|}{m_2}\left(\frac{\bar{c}\omega_3}{c^3}\frac{\breve{\rho}_3e^{\mu T}+3|\Delta|}{\overset{\frown}{\rho}_2\Delta^2}+\frac{\omega_2{}^2\Upsilon_2}{c^3}\right)\frac{e^{\mu T}}{\mu}\le\Upsilon_2;$$
$$\frac{|e_3e_1|}{m_3}\left(\frac{1}{c\overset{\frown}{\rho}_3{}^2}+\frac{\bar{c}\omega_2(\breve{\rho}_2+\breve{\rho}_3+3)}{c^3\Delta^2}+\frac{\omega_3{}^2R_3}{c^3}\right)\frac{e^{\mu T}}{\mu}\le R_3;$$
$$\phi_{30}+\frac{|e_3e_1|}{m_3c^3}\left[\frac{\beta\omega_2}{\cos(\pi/2-\delta)}\frac{\breve{\rho}_2+3c\Delta}{\overset{\frown}{\rho}_3c\Delta^2}+\omega_3{}^2\Phi_3\right]\frac{e^{\mu T}}{\mu}\le\Phi_3\ (\phi_{30}>0);$$
$$\frac{|e_3e_1|}{m_3c^3}\left(\frac{\bar{c}\omega_2}{c^3}\frac{\breve{\rho}_2e^{\mu T}+3\Delta}{\overset{\frown}{\rho}_3\Delta^2}+\frac{\omega_3{}^2\Upsilon_3}{c^3}\right)\frac{e^{\mu T}}{\mu}\le\Upsilon_3;$$

(2)
$$\left(1+\frac{e^{\mu T}-1}{\mu T}\right)\frac{|e_2e_1|}{m_2}\left(\frac{1}{c\overset{\frown}{\rho}_2{}^2}+\frac{\bar{c}\omega_3(\breve{\rho}_2+\breve{\rho}_3+3|\Delta|)}{c^3\Delta^2}+\frac{\omega_2{}^2R_2}{c^3}\right)\frac{e^{\mu T}}{\mu}\le\omega_2R_2;$$
$$\left(1+\frac{e^{\mu T}-1}{\mu T}\right)\frac{|e_2e_1|}{m_2c^3}\left[\frac{\bar{c}\omega_3\left(\breve{\rho}_3+3|\Delta|\right)}{c^2\overset{\frown}{\rho}_2\cos(\pi/2-\delta)\Delta^2}+\omega_2{}^2\Phi_2\right]\le\omega_2\Phi_2;$$
$$\left(1+\frac{e^{\mu T}-1}{\mu T}\right)\frac{|e_2e_1|}{m_2c^3}\left[\frac{\bar{c}\omega_3\left(\breve{\rho}_3+3|\Delta|\right)}{c^2\overset{\frown}{\rho}_2\Delta^2}+\omega_2{}^2\Upsilon_2\right]\le\omega_2\Upsilon_2;$$
$$\left(1+\frac{e^{\mu T}-1}{\mu T}\right)\frac{|e_3e_1|}{m_3c}\left[\frac{1}{\overset{\frown}{\rho}_3{}^2}+\frac{\bar{c}\omega_2\left(\breve{\rho}_2+\breve{\rho}_3+3|\Delta|\right)}{c^3\Delta^2}+\frac{\omega_3{}^2R_3}{c^2}\right]\le\omega_3R_3;$$
$$\left(1+\frac{e^{\mu T}-1}{\mu T}\right)\frac{|e_3e_1|}{m_3c^3}\left[\frac{\bar{c}\omega_2\left(\breve{\rho}_2+3|\Delta|\right)}{c^2\overset{\frown}{\rho}_3\cos(\pi/2-\delta)\Delta^2}+\omega_3{}^2\Phi_3\right]\le\omega_3\Phi_3;$$
$$\left(1+\frac{e^{\mu T}-1}{\mu T}\right)\frac{|e_3e_1|}{m_3c^3}\left[\frac{\bar{c}\omega_2\left(\breve{\rho}_2+3|\Delta|\right)}{c^2\overset{\frown}{\rho}_3\Delta^2}+\omega_3{}^2\Upsilon_3\right]\le\omega_3\Upsilon_3$$

then, there exists a unique T-periodic solution $(r_2,\phi_2,\eta_2,r_3,\phi_3,\eta_3)\in M_{r_2}\times M_{\phi_2}\times M_{\eta_2}\times M_{r_3}\times M_{\phi_3}\times M_{\eta_3}$ *of (BS).*

Principal remark. *Let us note that we have infinite number of inequalities in the theorem. The above inequalities imply that the operator functions and their first derivatives belong to* $M_{r_2}\times M_{\phi_2}\times M_{\eta_2}\times M_{r_3}\times M_{\phi_3}\times M_{\eta_3}$*. It is easy to check that for the third derivatives, we obtain respectively* $\omega_2{}^4$ *and* $\omega_2{}^3$ *and so on.*

Proof. First, we show that B maps the set $M_{r_2}\times M_{\phi_2}\times M_{\eta_2}\times M_{r_3}\times M_{\phi_3}\times M_{\eta_3}$ into itself. □

From Remark 4, we have $B_{r_2}^{(k)}(.)(T_k)=\frac{1}{2}\int\limits_{T_k}^{T_{k+1}}G_{r_2}(s)ds-\frac{1}{T}\int\limits_{T_k}^{T_{k+1}}\int\limits_{T_k}^{s}G_{r_2}(\theta)d\theta ds=0$, $(k=0,1,2,\dots)$ And since $\int\limits_{T_k}^{T_{k+1}}\left(\frac{t-T_k}{T}-\frac{1}{2}\right)dt=0$, we obtain

$$\int\limits_{T_k}^{T_{k+1}}B_{r_2}^{(k)}(.)(t)\,dt=\int\limits_{T_k}^{T_{k+1}}\int\limits_{T_k}^{t}G_{r_2}(s)ds\,dt-\int\limits_{T_k}^{T_{k+1}}\left(\frac{t-T_k}{T}-\frac{1}{2}\right)dt\int\limits_{T_k}^{T_{k+1}}G_{r_2}(s)ds-\int\limits_{T_k}^{T_{k+1}}\int\limits_{T_k}^{\theta}G_{r_2}(s)dsd\theta=0.$$

In addition, in view of $\rho\varphi\le\bar{c}$ and the inequalities from Remark 3, we have

$$\left|B_{r_2}^{(k)}(.)(t)\right| \le \int_{T_k}^{t} |G_{r_2}(s)|ds + \left|\frac{1}{2}\int_{T_k}^{T_{k+1}} G_{r_2}(s)ds + \frac{1}{T}\int_{T_k}^{T_{k+1}}\int_{T_k}^{t} G_{r_2}(s)dsdt\right| = \int_{T_k}^{t} |G_{r_2}(s)|ds + \left|\int_{T_k}^{T_{k+1}} G_{r_2}(s)ds\right| \le$$
$$\le e^{\mu(t-T_k)}\frac{|e_2e_1|}{m_2c}\left(\frac{1}{\widehat{\rho}_2{}^2} + \frac{\beta\omega_3(\breve{\rho}_2+\breve{\rho}_3+3\Delta)}{c^2\Delta^2} + \frac{\omega_2{}^2R_2}{c^2}\right)\left(\frac{1}{\mu} + \frac{e^{\mu T}-1}{\mu}\right) \le$$
$$\le e^{\mu(t-T_k)}\frac{|e_2e_1|}{m_2c}\left(\frac{1}{\widehat{\rho}_2{}^2} + \frac{\beta\omega_3(\breve{\rho}_2+\breve{\rho}_3+3\Delta)}{c^2\Delta^2} + \frac{\omega_2{}^2R_2}{c^2}\right)\frac{e^{\mu T}}{\mu} \le R_2e^{\mu(t-T_k)}.$$

For the second component of B, we have

$$B_{\phi_2}^{(k)}(.)(T_k) = \phi_{02} + \frac{1}{2}\int_{T_k}^{T_{k+1}} G_{\phi_2}(.)(s)ds - \frac{1}{T}\int_{T_k}^{T_{k+1}}\int_{T_k}^{s} G_{\phi_2}(.)(\theta)d\theta ds = \phi_{02},\ (k = 0,1,2,\ldots);$$
$$\int_{T_k}^{T_{k+1}} B_{\phi_2}^{(k)}(.)(t)\,dt = T\phi_{02} + \int_{T_k}^{T_{k+1}}\int_{T_k}^{t} G_{\phi_2}(s)ds - \int_{T_k}^{T_{k+1}}\left(\frac{t-T_k}{T} - \frac{1}{2}\right)dt\int_{T_k}^{T_{k+1}} G_{\phi_2}(s)ds - \int_{T_k}^{T_{k+1}}\int_{T_k}^{t} G_{\phi_2}(s)dsdt = T\phi_{02}$$

and

$$\left|B_{\phi_2}^{(k)}(.)(t)\right| \le \phi_{02} + \int_{T_k}^{t}|G_{\phi_2}(s)|ds + \left|\int_{T_k}^{T_{k+1}} G_{\phi_2}(s)ds\right| \le e^{\mu(t-T_k)}\left(\phi_{02} + \frac{|e_2e_3|}{m_2c^3}\left(\frac{\beta\omega_3}{\cos(\pi/2-\delta)}\frac{\breve{\rho}_3 + 3c\Delta}{\widehat{\rho}_2c\Delta^2} + \omega_2{}^2\Phi_2\right)\frac{e^{\mu T}}{\mu}\right) \le \Phi_2e^{\mu(t-T_k)}.$$

For the third component of B, we obtain

$$\left|B_{\eta_2}^{(k)}(.)(t)\right| \le \int_{T_k}^{t}|G_{\eta_2}(s)|ds + \left|\int_{T_k}^{T_{k+1}} G_{\eta_2}(s)ds\right| \le e^{\mu(t-T_k)}\frac{|e_2e_1|}{m_2}\left(\frac{\bar{c}\omega_3}{c^3}\frac{\breve{\rho}_3e^{\mu T} + 3|\Delta|}{\widehat{\rho}_2\Delta^2} + \frac{\omega_2{}^2\Upsilon_2}{c^3}\right)\frac{e^{\mu T}}{\mu} \le \Upsilon_2e^{\mu(t-T_k)}.$$

For the remaining three components of B, we have

$$\left|B_{r_3}^{(k)}(.)(t)\right| \le e^{\mu(t-T_k)}\frac{|e_3e_1|}{m_3c}\left(\frac{1}{\rho_3{}^2} + \frac{\beta\omega_2(\breve{\rho}_2+\breve{\rho}_3+3)}{c^2\Delta^2} + \frac{\omega_3{}^2R_3}{c^2}\right)\frac{e^{\mu T}}{\mu} \le R_3e^{\mu(t-T_k)};$$
$$\left|B_{\phi_3}^{(k)}(.)(t)\right| \le e^{\mu(t-T_k)}\phi_{03} + e^{\mu(t-T_k)}\frac{|e_3e_1|}{m_3c^3}\left[\frac{\beta\omega_2}{\cos(\pi/2-\delta)}\frac{\breve{\rho}_2+3c\Delta}{\widehat{\rho}_3c\Delta^2} + \omega_3{}^2\Phi_3\right]\frac{e^{\mu T}}{\mu} \le \Phi_3e^{\mu(t-T_k)};$$
$$\left|B_{\eta_3}^{(k)}(.)(t)\right| \le^{\mu(t-T_k)}\frac{|e_3e_1|}{m_3c^3}\left[\frac{\bar{c}\omega_2\left(\breve{\rho}_2+3|\Delta|\right)}{c^2\widehat{\rho}_3\Delta^2} + \omega_3{}^2\Upsilon_3\right]\frac{e^{\mu T}}{\mu} \le \Upsilon_3e^{\mu(t-T_k)}.$$

For the first-order derivatives of components of B, we obtain

$$\left|\frac{dB_{r_2}^{(k)}(.)(t)}{dt}\right| \le |G_{r_2}(.)(t)| + \frac{1}{T}\left|\int_{T_k}^{T_{k+1}} G_{r_2}(.)(s)ds\right| \le e^{\mu(t-T_k)}\left(1 + \frac{e^{\mu T}-1}{\mu T}\right)\frac{|e_2e_1|}{m_2c}\left[\frac{1}{\widehat{\rho}_2{}^2} + \frac{\bar{c}\omega_3\left(\breve{\rho}_2+\breve{\rho}_3+3\Delta\right)}{c^3\Delta^2} + \frac{\omega_2{}^2R_2}{c^2}\right] \le \omega_2R_2e^{\mu(t-T_k)};$$
$$\left|\frac{dB_{\phi_2}^{(k)}(.)(t)}{dt}\right| \le |G_{\phi_2}(.)(t)| + \frac{1}{T}\left|\int_{T_k}^{T_{k+1}} G_{\phi_2}(.)(s)ds\right| \le e^{\mu(t-T_k)}\left(1 + \frac{e^{\mu T}-1}{\mu T}\right)\frac{|e_2e_3|}{m_2c^3}\left[\frac{\bar{c}\omega_3\left(\breve{\rho}_3+3\Delta\right)}{c^2\widehat{\rho}_2\cos(\pi/2-\delta)\Delta^2} + \omega_2{}^2\Phi_2\right] \le \omega_2\Phi_2e^{\mu(t-T_k)};$$
$$\left|\frac{dB_{\eta_2}^{(k)}(.)(t)}{dt}\right| \le |G_{\eta_2}(.)(t)| + \frac{1}{T}\left|\int_{T_k}^{T_{k+1}} G_{\eta_2}(.)(s)ds\right| \le e^{\mu(t-T_k)}\left(1 + \frac{e^{\mu T}-1}{\mu T}\right)\frac{|e_2e_3|}{m_2c^3}\left[\frac{\bar{c}\omega_3\left(\breve{\rho}_3+3\Delta e^{\mu T}\right)}{c^2\widehat{\rho}_2\Delta^2} + \omega_2{}^2\Upsilon_2\right] \le \omega_2\Upsilon_2e^{\mu(t-T_k)};$$
$$\left|\frac{dB_{r_3}^{(k)}(.)(t)}{dt}\right| \le e^{\mu(t-T_k)}\left(1 + \frac{e^{\mu T}-1}{\mu T}\right)\frac{|e_3e_1|}{m_3c}\left[\frac{1}{\rho_3{}^2} + \frac{\bar{c}\omega_2\left(\breve{\rho}_2+\breve{\rho}_3+3\Delta\right)}{c^3\Delta^2} + \frac{\omega_3{}^2R_3}{c^2}\right] \le \omega_3R_3e^{\mu(t-T_k)};$$
$$\left|\frac{dB_{\phi_3}^{(k)}(.)(t)}{dt}\right| \le e^{\mu(t-kT)}\left(1 + \frac{e^{\mu T}-1}{\mu T}\right)\frac{|e_2e_3|}{m_3c^3}\left[\frac{\bar{c}\omega_2\left(\breve{\rho}_2+3\Delta\right)}{c^2\widehat{\rho}_3\cos(\pi/2-\delta)\Delta^2} + \omega_3{}^2\Phi_3\right] \le \omega_3\Phi_3e^{\mu(t-T_k)};$$
$$\left|\frac{dB_{\eta_3}^{(k)}(.)(t)}{dt}\right| \le e^{\mu(t-T_k)}\frac{|e_2e_3|}{m_3c^3}\left[\frac{\bar{c}\omega_2\left(\breve{\rho}_2+3\Delta\right)}{c^2\widehat{\rho}_3\Delta^2} + \omega_3{}^2\Upsilon_3\right] \le \omega_3\Upsilon_3e^{\mu(t-T_k)}.$$

The estimates for the higher-order derivatives of the components of B are obtained in Appendix D.

In what follows, we show that operator B is a contractive one in the sense of [8].

In view of Remark 2, the retarded functions act in the following way:

$$t-\tau_{23}(t):[T_k,T_{k+1}]\to[T_{k-1},T_k],\ t-\tau_{32}(t):[T_k,T_{k+1}]\to[T_{k-1},T_k].$$

The index set of $M_{r_2}\times M_{\phi_2}\times M_{\eta_2}\times M_{r_3}\times M_{\phi_3}\times M_{\eta_3}$ is (k,m), where $k,m=0,1,2,\ldots$ We introduce the following mappings of the index set into itself: $j_1(k,m)=(k,m), j_2(k,m)=(k-1,m+1)$, $j_3(k,m)=(k,m+2)$.

Denote by $\partial\overline{G}_{r_2}=\max\left\{\frac{\partial\overline{G}_{r_2}}{\partial\rho_2},\frac{\partial\overline{G}_{r_2}}{\partial\varphi_2},\frac{\partial\overline{G}_{r_2}}{\partial\lambda_2},\frac{\partial\overline{G}_{r_2}}{\partial\rho_3},\frac{\partial\overline{G}_{r_2}}{\partial\varphi_3},\frac{\partial\overline{G}_{r_2}}{\partial\lambda_3},\frac{\partial\overline{G}_{r_2}}{\partial\dot{r}_3},\frac{\partial\overline{G}_{r_2}}{\partial\dot{\phi}_3},\frac{\partial\overline{G}_{r_2}}{\partial\dot{\eta}_3},\frac{\partial\overline{G}_{r_2}}{\partial\ddot{r}_2}\right\}$ (cf. Appendix E). Then,

$$\left|B_{r_2}(r_2,\phi_2,\eta_2,r_3,\phi_3,\eta_3)-B_{r_2}(\overline{r}_2,\overline{\phi}_2,\overline{\eta}_2,\overline{r}_3,\overline{\phi}_3,\overline{\eta}_3)\right|\le$$

$$\le\frac{\partial\overline{G}_{r_2}}{\partial\rho_2}\int_{T_k}^{t}|\rho_2(s)-\overline{\rho}_2(s)|ds+\frac{\partial\overline{G}_{r_2}}{\partial\varphi_2}\int_{T_k}^{t}|\varphi_2(s)-\overline{\varphi}_2(s)|ds+\frac{\partial\overline{G}_{r_2}}{\partial\lambda_2}\int_{T_k}^{t}\left|\lambda_2(s)-\overline{\lambda}_2(s)\right|ds+\frac{\partial\overline{G}_{r_2}}{\partial\rho_3}\int_{T_k}^{t}|\rho_3(s-\tau_{23}(s))-\overline{\rho}_3(s-\tau_{23}(s))|ds+$$

$$+\frac{\partial\overline{G}_{r_2}}{\partial\varphi_3}\int_{T_k}^{t}|\varphi_3(s-\tau_{23}(s))-\overline{\varphi}_3(s-\tau_{23}(s))|ds+\frac{\partial\overline{G}_{r_2}}{\partial\lambda_3}\int_{T_k}^{t}\left|\lambda_3(s-\tau_{23}(s))-\overline{\lambda}_3(s-\tau_{23}(s))\right|ds+\frac{\partial\overline{G}_{r_2}}{\partial\dot{r}_3}\int_{T_k}^{t}\left|\dot{r}_3(s-\tau_{23}(s))-\dot{\overline{r}}_3(s-\tau_{23}(s))\right|ds+$$

$$+\frac{\partial\overline{G}_{r_2}}{\partial\dot{\phi}_3}\int_{T_k}^{t}\left|\dot{\phi}_{\frown 3}(s-\tau_{23}(s))-\dot{\overline{\phi}}_3(s-\tau_{23}(s))\right|ds+\frac{\partial\overline{G}_{r_2}}{\partial\dot{\eta}_3}\int_{T_k}^{t}\left|\dot{\eta}_3(s-\tau_{23}(s))-\dot{\overline{\eta}}_3(s-\tau_{23}(s))\right|ds+\frac{\partial\overline{G}_{r_2}}{\partial\ddot{r}_2}\int_{T_k}^{t}\left|\ddot{r}_2(s)-\ddot{\overline{r}}_2(s)\right|ds\le$$

$$\le\frac{\partial\overline{G}_{r_2}}{\partial\rho_2}\int_{T_k}^{t}\int_{T_k}^{s}|r_2(p)-\overline{r}_2(p)|e^{-\mu(p-T_k)}e^{\mu(p-T_k)}dpds+\frac{\partial\overline{G}_{r_2}}{\partial\varphi_2}\int_{T_k}^{t}\int_{T_k}^{s}\left|\phi_2(p)-\overline{\phi}_2(p)\right|e^{-\mu(p-T_k)}e^{\mu(p-T_k)}dpds+$$

$$+\frac{\partial\overline{G}_{r_2}}{\partial\lambda_2}\int_{T_k}^{t}\int_{T_k}^{s}|\eta_2(p)-\overline{\eta}_2(p)|e^{-\mu(p-T_k)}e^{\mu(p-T_k)}dpds+\frac{\partial\overline{G}_{r_2}}{\partial\rho_3}\int_{T_k}^{t}\int_{T_k}^{s-\tau_{23}(s)}|r_3(p)-\overline{r}_3(p)|e^{-\mu(p-T_k)}e^{\mu(p-T_k)}dpds+$$

$$+\frac{\partial\overline{G}_{r_2}}{\partial\varphi_3}\int_{T_k}^{t}\int_{T_k}^{s-\tau_{23}(s)}\left|\phi_3(p)-\overline{\phi}_3(p)\right|e^{-\mu(p-T_k)}e^{\mu(p-T_k)}dpds+\frac{\partial\overline{G}_{r_2}}{\partial\lambda_3}\int_{T_k}^{t}\int_{T_k}^{s-\tau_{23}(s)}|\eta_2(p)-\overline{\eta}_2(p)|e^{-\mu(p-T_k)}e^{\mu(p-T_k)}dpds+$$

$$+\frac{\partial\overline{G}_{r_2}}{\partial\dot{r}_3}\int_{T_k}^{t}\left|\dot{r}_3(s-\tau_{23}(s))-\dot{\overline{r}}_3(s-\tau_{23}(s))\right|e^{-\mu(s-\tau_{23}(s)-T_k)}e^{\mu(s-\tau_{23}(s)-T_k)}ds+\frac{\partial\overline{G}_{r_2}}{\partial\dot{\phi}_3}\int_{T_k}^{t}\left|\dot{\phi}_{\frown 3}(s-\tau_{23}(s))-\dot{\overline{\phi}}_3(s-\tau_{23}(s))\right|e^{-\mu(s-\tau_{23}(s)-T_k)}e^{\mu(s-\tau_{23}(s)-T_k)}ds+$$

$$+\frac{\partial\overline{G}_{r_2}}{\partial\dot{\eta}_3}\int_{T_k}^{t}\left|\dot{\eta}_3(s-\tau_{23}(s))-\dot{\overline{\eta}}_3(s-\tau_{23}(s))\right|e^{-\mu(s-\tau_{23}(s)-T_k)}e^{\mu(s-\tau_{23}(s)-T_k)}ds+\frac{\partial\overline{G}_{r_2}}{\partial\ddot{r}_2}\int_{T_k}^{t}\left|\ddot{r}_2(s)-\ddot{\overline{r}}_2(s)\right|e^{-\mu(s-T_k)}e^{\mu(s-T_k)}ds\le$$

$$\le\partial\overline{G}_{r_2}e^{\mu(t-T_k)}\left[\frac{d_{(k,0)}(r_2,\overline{r}_2)}{\mu^2}+\frac{d_{(k,0)}\left(\phi_2,\overline{\phi}_2\right)}{\mu^2}+\frac{d_{(k,0)}(\eta_2,\overline{\eta}_2)}{\mu^2}+\left(\frac{d_{(k,0)}(r_3,\overline{r}_3)}{\mu}+\frac{d_{(k,0)}\left(\phi_3,\overline{\phi}_3\right)}{\mu}+\frac{d_{(k,0)}\left(\eta_3,\overline{\eta}_3\right)}{\mu}\right)\left|\int_{T_k}^{t}\left(e^{\mu(s-\tau_{23}-T_k)}-1\right)ds\right|+\right.$$

$$\left.+\left(\frac{\omega d_{(k-1,1)}(r_3,\overline{r}_3)}{\mu}+\frac{\omega d_{(k-1,1)}\left(\phi_3,\overline{\phi}_3\right)}{\mu}+\frac{\omega d_{(k-1,1)}\left(\eta_3,\overline{\eta}_3\right)}{\mu}\right)\int_{T_k}^{t}e^{\mu(s-\tau_{23}(s)-T_k)}ds+\frac{\omega^2 d_{(k,2)}(r_2,\overline{r}_2)}{\mu}\int_{T_k}^{t}e^{\mu(s-T_k)}ds\right]\le$$

$$\le\partial\overline{G}_{r_2}\left[e^{\mu(t-T_k)}\left(\frac{d_{(k,0)}(r_2,\overline{r}_2)}{\mu^2}+\frac{d_{(k,0)}\left(\phi_2,\overline{\phi}_2\right)}{\mu^2}+\frac{d_{(k,0)}(\eta_2,\overline{\eta}_2)}{\mu^2}\right)+\left(\frac{d_{(k,0)}(r_3,\overline{r}_3)}{\mu}+\frac{d_{(k,0)}\left(\phi_3,\overline{\phi}_3\right)}{\mu}+\frac{d_{(k,0)}\left(\eta_3,\overline{\eta}_3\right)}{\mu}\right)\left|\int_{T_k}^{t}e^{\mu(s-T_k)}(1-e^{-\mu\tau_{23}})ds\right|+\right.$$

$$\left.+e^{\mu(t-T_k)}\frac{\omega}{\mu^2}\left(d_{(k-1,1)}(r_3,\overline{r}_3)+d_{(k-1,1)}\left(\phi_3,\overline{\phi}_3\right)+d_{(k-1,1)}(\eta_3,\overline{\eta}_3)\right)+e^{\mu(t-T_k)}\frac{\omega^2 d_{(k,2)}(r_2,\overline{r}_2)}{\mu^2}\right]\le$$

$$\le\partial\overline{G}_{r_2}e^{\mu(t-T_k)}\left[2\frac{d_{(k,0)}(r_2,\overline{r}_2)+d_{(k,0)}\left(\phi_2,\overline{\phi}_2\right)+d_{(k,0)}(\eta_2,\overline{\eta}_2)}{\mu^2}+\frac{\omega\left(d_{(k-1,1)}(r_3,\overline{r}_3)+d_{(k-1,1)}\left(\phi_3,\overline{\phi}_3\right)+d_{(k-1,1)}\left(\eta_3,\overline{\eta}_3\right)\right)}{\mu^2}+\frac{\omega^2 d_{(k,2)}(r_2,\overline{r}_2)}{\mu^2}\right].$$

It follows that

$$d_{(k,0)}\left(B_{r_2}^{(k)}(.),\overline{B}_{r_2}^{(k)}(.)\right)\le$$
$$\le\partial\overline{G}_{r_2}\left[\frac{2}{\mu^2}d_{(k,0)}\left((r_2,\phi_2,\eta_2,r_3,\phi_3,\eta_3),(\overline{r}_2,\overline{\phi}_2,\overline{\eta}_2,\overline{r}_3,\overline{\phi}_3,\overline{\eta}_3)\right)+\frac{\omega}{\mu^2}d_{(k-1,1)}\left((r_2,\phi_2,\eta_2,r_3,\phi_3,\eta_3),(\overline{r}_2,\overline{\phi}_2,\overline{\eta}_2,\overline{r}_3,\overline{\phi}_3,\overline{\eta}_3)\right)+\right.$$
$$\left.+\frac{\omega^2}{\mu^2}d_{(k,2)}\left((r_2,\phi_2,\eta_2,r_3,\phi_3,\eta_3),(\overline{r}_2,\overline{\phi}_2,\overline{\eta}_2,\overline{r}_3,\overline{\phi}_3,\overline{\eta}_3)\right)\right].$$

In a similar way, one obtains

$$d_{(k,0)}\left(B_{r_3}^{(k)}(.),\overline{B}_{r_3}^{(k)}(.)\right)\leq$$
$$\leq\partial\overline{G}_{r_3}\left[\tfrac{2}{\mu^2}d_{(k,0)}\left((r_2,\phi_2,\eta_2,r_3,\phi_3,\eta_3),(\overline{r}_2,\overline{\phi}_2,\overline{\eta}_2,\overline{r}_3,\overline{\phi}_3,\overline{\eta}_3)\right)+\tfrac{\omega}{\mu^2}d_{(k-1,1)}\left((r_2,\phi_2,\eta_2,r_3,\phi_3,\eta_3),(\overline{r}_2,\overline{\phi}_2,\overline{\eta}_2,\overline{r}_3,\overline{\phi}_3,\overline{\eta}_3)\right)+\right.$$
$$\left.+\tfrac{\omega^2}{\mu^2}d_{(k,2)}\left((r_2,\phi_2,\eta_2,r_3,\phi_3,\eta_3),(\overline{r}_2,\overline{\phi}_2,\overline{\eta}_2,\overline{r}_3,\overline{\phi}_3,\overline{\eta}_3)\right)\right].$$

Further on, we have

$$\left|B_{\phi_2}(r_2,\phi_2,\eta_2,r_3,\phi_3,\eta_3)-B_{\phi_2}(\overline{r}_2,\overline{\phi}_2,\overline{\eta}_2,\overline{r}_3,\overline{\phi}_3,\overline{\eta}_3)\right|\leq$$
$$\leq\frac{\partial\overline{G}_{\phi_2}}{\partial\rho_2}\int_{T_k}^{t}|\rho_2(s)-\overline{\rho}_2(s)|ds+\frac{\partial\overline{G}_{\phi_2}}{\partial\varphi_2}\int_{T_k}^{t}|\varphi_2(s)-\overline{\varphi}_2(s)|ds+\frac{\partial\overline{G}_{\phi_2}}{\partial\lambda_2}\int_{T_k}^{t}\left|\lambda_2(s)-\overline{\lambda}_2(s)\right|ds+\frac{\partial\overline{G}_{\phi_2}}{\partial\rho_3}\int_{T_k}^{t}\left|\rho_3(s-\tau_{23}(s))-\overline{\rho}_3(s-\tau_{23}(s))\right|ds+$$
$$\frac{\partial\overline{G}_{\phi_2}}{\partial\varphi_3}\int_{T_k}^{t}\left|\varphi_3(s-\tau_{23}(s))-\overline{\varphi}_3(s-\tau_{23}(s))\right|ds+\frac{\partial\overline{G}_{\phi_2}}{\partial\lambda_3}\int_{T_k}^{t}\left|\lambda_3(s-\tau_{23}(s))-\overline{\lambda}_3(s-\tau_{23}(s))\right|ds+\frac{\partial\overline{G}_{\phi_2}}{\partial\dot{r}_3}\int_{T_k}^{t}\left|\dot{r}_3(s-\tau_{23}(s))-\dot{\overline{r}}_3(s-\tau_{23}(s))\right|ds+$$
$$\frac{\partial\overline{G}_{\phi_2}}{\partial\dot{\phi}_3}\int_{T_k}^{t}\left|\dot{\phi}_{\frown 3}(s-\tau_{23}(s))-\dot{\overline{\phi}}_3(s-\tau_{23}(s))\right|ds+\frac{\partial\overline{G}_{\phi_2}}{\partial\dot{\eta}_3}\int_{T_k}^{t}\left|\dot{\eta}_3(s-\tau_{23}(s))-\dot{\overline{\eta}}_3(s-\tau_{23}(s))\right|ds+\frac{\partial\overline{G}_{\phi_2}}{\partial\ddot{r}_2}\int_{T_k}^{t}\left|\ddot{\phi}_2(s)-\ddot{\overline{\phi}}_2(s)\right|ds\leq$$
$$\left|\frac{\partial G_{\phi_2}}{\partial\rho_2}\right|\leq\frac{|e_2e_3|\overline{c}}{m_2\cos(\pi/2-\delta)}\left(\frac{c^3\breve{\rho}_3}{(\Delta-(R_3/\mu)e^{\mu T})^3}\frac{2\omega}{\widehat{\rho}_2{}^2}+\frac{6c}{c^4\Delta-(R_3/\mu)e^{\mu T})\widehat{\rho}_2{}^2}+\frac{c\omega}{c^4\Delta-(R_3/\mu)e^{\mu T})\widehat{\rho}_2}\right)=\frac{\partial\overline{G}_{\phi_2}}{\partial\rho_2}$$

and so on.

Denoting by $\overline{B}_{r_2}=B_{r_2}(\overline{r}_2,\overline{\phi}_2,\overline{\eta}_2,\overline{r}_3,\overline{\phi}_3,\overline{\eta}_3),\ldots,\overline{B}_{\eta_3}=B_{\eta_3}(\overline{r}_2,\overline{\phi}_2,\overline{\eta}_2,\overline{r}_3,\overline{\phi}_3,\overline{\eta}_3)$, we obtain

$$d_{(k,0)}\left((B_{r_2},B_{\phi_2},B_{\eta_3},B_{r_3},B_{\phi_3},B_{\eta_3}),(\overline{B}_{r_2},\overline{B}_{\phi_2},\overline{B}_{\eta_3},\overline{B}_{r_3},\overline{B}_{\phi_3},\overline{B}_{\eta_3})\right)\leq$$
$$\leq\tfrac{2(\partial\overline{G}_{r_3}+\ldots+\partial\overline{G}_{\eta_3})}{\mu^2}d_{(k,0)}\left((r_2,\phi_2,\eta_2,r_3,\phi_3,\eta_3),(\overline{r}_2,\overline{\phi}_2,\overline{\eta}_2,\overline{r}_3,\overline{\phi}_3,\overline{\eta}_3)\right)+\tfrac{\omega(\partial\overline{G}_{r_3}+\ldots+\partial\overline{G}_{\eta_3})}{\mu^2}d_{(k-1,1)}\left((r_2,\phi_2,\eta_2,r_3,\phi_3,\eta_3),(\overline{r}_2,\overline{\phi}_2,\overline{\eta}_2,\overline{r}_3,\overline{\phi}_3,\overline{\eta}_3)\right)+$$
$$+\tfrac{\omega^2(\partial\overline{G}_{r_3}+\ldots+\partial\overline{G}_{\eta_3})}{\mu^2}d_{(k,2)}\left((r_2,\phi_2,\eta_2,r_3,\phi_3,\eta_3),(\overline{r}_2,\overline{\phi}_2,\overline{\eta}_2,\overline{r}_3,\overline{\phi}_3,\overline{\eta}_3)\right).$$

If $\frac{2(\partial\overline{G}_{r_3}+\ldots+\partial\overline{G}_{\eta_3})}{\mu^2}<\frac{1}{3}$; $\frac{\omega(\partial\overline{G}_{r_3}+\ldots+\partial\overline{G}_{\eta_3})}{\mu^2}<\frac{1}{3}$; $\frac{\omega^2(\partial\overline{G}_{r_3}+\ldots+\partial\overline{G}_{\eta_3})}{\mu^2}<\frac{1}{3}$ are not satisfied, then we apply the inequalities from Appendix A and obtain the desired inequalities for a sufficiently large $m\in N$:

$$\frac{\omega^m}{\mu^m}\frac{2(\partial\overline{G}_{r_3}+\ldots+\partial\overline{G}_{\eta_3})}{\mu^2}<\frac{1}{3};\ \frac{\omega^m}{\mu^m}\frac{\omega(\partial\overline{G}_{r_3}+\ldots+\partial\overline{G}_{\eta_3})}{\mu^2}<\frac{1}{3};\ \frac{\omega^m}{\mu^m}\frac{\omega^2(\partial\overline{G}_{r_3}+\ldots+\partial\overline{G}_{\eta_3})}{\mu^2}<\frac{1}{3}.$$

It is easy to see that the space $M_{r_2}\times M_{\phi_2}\times M_{\eta_2}\times M_{r_3}\times M_{\phi_3}\times M_{\eta_3}$ is $(j_1,j_2,j_3)-$ bounded and the maps $j_s: A\to A$ $(s=1,2,3)$ are commutative. Then, the operator $(B_{2r},B_{2\phi},B_{3r},B_{3\phi})$, in view of the fixed point theorem from [8], has a unique fixed point, which, in view of the Main Lemma, is a T-periodic solution of (BS).

Theorem 1 is thus proved.

Numerical results

We check the inequalities from the Main Theorem:

$$\frac{|e_2e_1|}{m_2}\left(\frac{1}{c\widehat{\rho}_2{}^2}+\frac{\overline{c}\omega_3(\breve{\rho}_2+\breve{\rho}_3+3\Delta)}{c^3\Delta^2}+\frac{\omega_2{}^2R_2}{c^3}\right)\frac{e^{\mu T}}{\mu}\leq R_2;\ \phi_{02}+\frac{|e_2e_3|}{m_2}\left(\frac{\overline{c}\omega_3(\breve{\rho}_3+3\Delta)}{\widehat{\rho}_2c^3\Delta^2\cos(\pi/2-\delta)}+\frac{\omega_2{}^2\Phi_2}{c^3}\right)\frac{e^{\mu T}}{\mu}\leq\Phi_2;$$
$$\frac{|e_2e_3|}{m_2}\left(\frac{\overline{c}\omega_3}{c^3}\frac{\breve{\rho}_3e^{\mu T}+3\Delta}{\widehat{\rho}_2\Delta^2}+\frac{\omega_2{}^2Y_2}{c^3}\right)\frac{e^{\mu T}}{\mu}\leq Y_2.$$

It is known (cf. [21]) that
$e_n=1.6\times10^{-19}C$; $m_n=9.11\times10^{-19}kg\ (n=2,3)$; $c=3\times10^8m/\sec$; $\breve{\rho}_n\approx\widehat{\rho}_n=5.3\times10^{-11}m$;
$\omega_2=\omega_3=4\times10^{16}$, $T=\frac{2\pi}{\omega}=\frac{6.28}{4\times10^{16}}\approx1.57\times10^{-16}$, $\Delta\approx c$, $\delta=\frac{\pi}{6}$, $\phi_{20}<\Phi_0$.

Choosing $\mu=6\times10^{16}>\omega=4\times10^{16}$ and $\mu T=6\times10^6\times1,.57\times10^{-16}=9.42>\omega T=2\pi$, one obtains $e^{9.42}\approx1.2\times10^4$, $\frac{e^{\mu T}}{\mu}=\frac{1.2\times10^4}{6\times10^{16}}\approx2\times10^{-13}$.

We have $(R_2+R_3)\frac{e^{\mu T}}{\mu} \le \frac{2\bar{c}}{\mu} = \frac{2.2\times10^6}{6\times10^{16}} \approx 0 \Rightarrow \Delta \approx |\rho_{02}-\rho_{03}|$;

$$2.81\times10^{-8}\left(\frac{1}{3\times10^8\times5.3\times10^{-11}} + \frac{4\times10^{16}(2\times5.3\times10^{-11}+3|\rho_{02}-\rho_{03}|)}{137\times9\times10^{16}|\rho_{02}-\rho_{03}|^2} + \frac{16\times10^{32}}{27\times10^{24}}R_2\right)2\times10^{-13} \le R_2;$$

$$5.62\times10^{-21}\left(\frac{4\times10^{16}(5.3\times10^{-11}+3|\rho_{02}-\rho_{03}|)}{137\times5.3\times10^{-11}\times9\times10^{16}|\rho_{02}-\rho_{03}|^2\times0.5} + \frac{4^2\times10^{32}}{27\times10^{24}}\Phi_2\right) \le \Phi_2-\phi_{02};$$

$$2.81\times10^{-8}\left(\frac{4\times10^{16}}{137\times9\times10^{16}}\frac{5.3\times10^{-11}\times4.8+3|\rho_{02}-\rho_{03}|}{5.3\times10^{-11}|\rho_{02}-\rho_{03}|^2} + \frac{16\times10^{32}}{27\times10^{24}}Y_2\right)2\times10^{-13} \le Y_2.$$

Since $|\rho_{02}-\rho_{03}| \approx 10^{-11}$, then

$$5.62\left(\frac{10^3}{15.9} + \frac{1}{4.11\times10^{12}\times10^{-22}} + \frac{1}{4.11\times10^{18}\times10^{-11}} + 6\times10^7 R_2\right)10^{-21} \le R_2.$$

For the second inequality, we obtain

$$5.62\times10^{-21}\left(6.5\times10^{19} + 3.7\times10^{19} + 0.6\times10^8\Phi_2\right) \le \Phi_2-\phi_{02}.$$

Finally, $5.62(1.6\times10^{20} + 1.8\times10^{19} + 6.10^7 Y_2)10^{-21} \le Y_2$.

7. Discussion on the Bohr and Sommerfeld Hypotheses

In accordance with Bohr's first hypothesis (cf. [22,23]) both electrons of the He atom move in circle orbits in a plane on opposite sides. This means that we have the Kepler problem with the conditions $\lambda_2 = 0,\ \lambda_3 = 0,\ \varphi_3(t) = \varphi_2(t)+\pi$, which yield

$$\varphi_2(t)-\varphi_2(t-\tau_{23}) \approx \dot\phi_2\tau_{23} \approx \omega\frac{\rho_{23}}{2c} \approx 10^{16}\frac{5.3\times10^{-11}}{6\times10^8} \approx 10^{-3};$$

$$\varphi_3(t-\tau_{32})-\varphi_3(t) \approx \dot\phi_3\tau_{32} \approx \omega\frac{\rho_{23}}{2c} \approx 10^{16}\frac{5.3\times10^{-11}}{6\times10^8} \approx 10^{-3}; \text{andthen,}$$

$$\cos\varphi_{23} = \cos(\varphi_2(t)-\varphi_3(t-\tau_{23})) = \cos(\varphi_2(t)-\varphi_2(t-\tau_{23})-\pi) = -\cos(\varphi_2(t)-\varphi_2(t-\tau_{23})) \simeq -1;$$

$$\sin\varphi_{23} = -\sin(\varphi_2-\varphi_3(t-\tau_{23})) = -\sin(\varphi_2(t)-\varphi_2(t-\tau_{23})-\pi) = \sin(\varphi_2(t)-\varphi_2(t-\tau_{23})) \simeq 0.$$

Since $\left\langle \overset{\rightarrow}{\xi}^{(23)}, \overset{\cdot}{\vec{u}}^{(3)} \right\rangle = \rho_2[\cos(\varphi_2-\varphi_3)-1]\ddot\rho_3 = -2\rho_2\ddot\rho_3$, $\left\langle \overset{\rightarrow}{\xi}^{(32)}, \overset{\cdot}{\vec{u}}^{(2)} \right\rangle = -2\rho_3\ddot\rho_2$, the equations of motion become

$$\dot r_2 = \frac{e_2e_1}{m_2}\frac{1}{c\rho_2{}^2} + \frac{e_2e_3}{m_2}\left(\frac{4\rho_2{}^2\dot r_3}{c^6\tau_{23}{}^3} + \frac{\dot r_3}{c^4\tau_{23}}\right) - \frac{e_2^2}{m_2}\frac{\ddot r_2}{c^3};\ \dot\phi_2 = -\frac{e_2e_3}{m_2}\left(\frac{\rho_3\dot\phi_3}{c^4\tau_{23}\rho_2} + \frac{\ddot\phi_2}{c^3}\right);$$

$$\dot r_3 = \frac{e_3e_1}{m_3}\frac{1}{c\rho_3{}^2} - \frac{e_2e_3}{m_3}\left(\frac{4\rho_3{}^2\ddot\rho_2}{c^6\tau_{32}{}^3} + \frac{\dot r_2}{c^4\tau_{23}}\right) - \frac{e_3^2}{m_3}\frac{\ddot r_3}{c^3};\ \dot\phi_3 = \frac{e_3e_2}{m_3}\left(\frac{\rho_2\dot\phi_2\cos\varphi_{32}}{c^4\tau_{32}\rho_3} - \frac{\ddot\phi_3}{c^3}\right).$$

This system is a particular case of (BS) and has a unique periodic solution.

The second Bohr hypothesis (cf. [22,23]) is as follows: both electrons move in two different planes at an angle of 60 degrees. This means that one can choose $\lambda_2 = 0,\ \lambda_3 = \pi/3 \Rightarrow \eta_3 = 0,\ \varphi_3(t) = \varphi_2(t)$ $+\alpha > 0$, where α will be defined below.

From (BS), the system of equations of motion is

$$\begin{aligned}
\dot r_2 &= \frac{e_2e_1}{m_2}\frac{1}{c\rho_2{}^2} - \frac{e_2e_3}{m_2}\frac{\rho_2-\rho_3\cos\varphi_{23}}{c^3}\frac{\left\langle \overset{\rightarrow}{\xi}^{(23)}, \overset{\cdot}{\vec{u}}^{(3)} \right\rangle}{c^3\tau_{23}{}^3} - \frac{e_2e_3}{m_2}\frac{\dot r_3\cos\varphi_{23}+\rho_3\dot\phi_3\sin\varphi_{23}}{c^4\tau_{23}} - \frac{e_2^2}{m_2}\frac{\ddot r_2}{c^3};\\
\dot\phi_2 &= \frac{e_2e_3}{m_2}\left[\frac{\rho_3\sin\varphi_{23}}{\tau_{23}{}^3\rho_2}\frac{\left\langle \overset{\rightarrow}{\xi}^{(23)}, \overset{\cdot}{\vec{u}}^{(3)} \right\rangle}{c^6} + \frac{\dot r_3\sin\varphi_{23}+\rho_3\dot\phi_3\cos\varphi_{23}}{c^4\tau_{23}\rho_2} - \frac{\ddot\phi_2}{c^3}\right];\\
\dot r_3 &= \frac{e_3e_1}{m_3}\frac{1}{c\rho_3{}^2} - \frac{e_2e_3}{m_3}\frac{\rho_3-\rho_2\cos\varphi_{32}}{c^3}\frac{\left\langle \overset{\rightarrow}{\xi}^{(32)}, \overset{\cdot}{\vec{u}}^{(2)} \right\rangle}{c^3\tau_{32}{}^3} - \frac{e_3e_2}{m_3}\frac{\dot r_2\cos\varphi_{32}+\rho_2\dot\phi_2\sin\varphi_{32}}{c^4\tau_{23}} - \frac{e_3^2}{m_3}\frac{\ddot r_3}{c^3};\\
\dot\phi_3 &= \frac{e_3e_2}{m_3}\left[\frac{\rho_2\sin\varphi_{32}}{\tau_{32}{}^3\rho_3}\frac{\left\langle \overset{\rightarrow}{\xi}^{(32)}, \overset{\cdot}{\vec{u}}^{(2)} \right\rangle}{c^6} + \frac{\dot r_2\sin\varphi_{32}+\rho_2\dot\phi_2\cos\varphi_{32}}{c^4\tau_{32}\rho_3} - \frac{\ddot\phi_3}{c^3}\right];\\
0 &= \frac{\rho_2\cos\varphi_{32}}{\tau_{32}{}^2}\frac{(\rho_3\cos\varphi_{32}-2\rho_2)\dot r_2+\sin\varphi_{32}\rho_2\rho_3\dot\phi_2}{2c^6} + \frac{\dot r_3\cos\varphi_{32}+\rho_2\dot\phi_2\sin\varphi_{32}}{c^4}.
\end{aligned} \tag{11}$$

Equation (11) can be considered as an additional condition for the existence of a periodic solution. Indeed,

$$\begin{array}{l}\varphi_{23} = \varphi_2(t) - \varphi_3(t - \tau_{23}),\ \varphi_3(t) = \varphi_2(t) + \alpha, \varphi_{23} = \varphi_2(t) - \varphi_2(t - \tau_{23}) - \alpha \approx -\alpha; \\ \varphi_{32} = \varphi_2(t - \tau_{32}) - \varphi_3(t),\ \varphi_3(t) = \varphi_2(t) + \alpha, \varphi_{32} = \varphi_2(t - \tau_{32}) - \varphi_2(t) - \alpha \approx -\alpha; \\ \cos\varphi_{32} \approx \cos\alpha, \sin\varphi_{32} \approx -\sin\alpha.\end{array}$$

Then, Equation (11) becomes

$$0 = \frac{\rho_2 \cos\alpha}{{\tau_{32}}^2} \frac{(\rho_3\cos\alpha - 2\rho_2)\dot{r}_2 + \rho_2\rho_3\dot{\phi}_2 \sin\alpha}{2c^6} + \frac{\dot{r}_3\cos\alpha + \rho_2\dot{\phi}_2\sin\alpha}{c^4}.$$

For $\rho_2 = \rho_3 = \rho = const.$, one obtains

$$\left(\frac{\cos\alpha}{{\tau_{32}}^2}\frac{\rho^2}{2c^2} + 1\right)\frac{\dot{\phi}_2\sin\alpha}{c^4} = 0 \Rightarrow \sin\alpha = 0 \Rightarrow \alpha = 0 \vee \alpha = \pi,$$

which confirms the Bohr hypothesis for $\alpha = \pi$. Then, the equations of motion become

$$\begin{array}{l}\dot{r}_2 = \frac{e_2e_1}{m_2}\frac{1}{c{\rho_2}^2} - \frac{e_2e_3}{m_2}\frac{\rho_2}{c^3}\frac{(\rho_2+2\rho_3)\dot{r}_3}{2c^3{\tau_{23}}^3} - \frac{e_2e_3}{m_2}\frac{\dot{r}_3}{2c^4\tau_{23}} - \frac{e_2^2}{m_2}\frac{\ddot{r}_2}{c^3};\ \dot{\phi}_2 = -\frac{e_2e_3}{m_2}\left(\frac{2\rho_3\dot{\phi}_3}{c^4\tau_{23}\rho_2} + \frac{\ddot{\phi}_2}{c^3}\right); \\ \dot{r}_3 = \frac{e_3e_1}{m_3}\frac{1}{c{\rho_3}^2} + \frac{e_2e_3}{m_3}\frac{(\rho_3+2\rho_2)\dot{r}_2}{2c^3{\tau_{32}}^3} + \frac{e_3e_2}{m_3}\frac{\dot{r}_2}{2c^4\tau_{23}} - \frac{e_3^2}{m_3}\frac{\ddot{r}_3}{c^3};\ \dot{\phi}_3 = -\frac{e_3e_2}{m_3}\left(\frac{2\rho_2\dot{\phi}_2}{c^4\tau_{32}\rho_3} + \frac{\ddot{\phi}_3}{c^3}\right).\end{array}$$

Later, additional assumptions are made by A. Sommerfeld [23], but they turn out, again, to be a particular case of our equations. We notice that the spin equation for the plane Kepler problem (investigated in [24]) has solutions with opposite signs.

8. Conclusions

We have obtained conditions for the existence–uniqueness of a periodic solution for the three-body problem of classical electrodynamics. This includes the He atom and the He-like atoms—for instance, the Li atom without one electron. We give a new formulation of the retarded action between moving particles, splitting the equations of motion. As in previous papers, we use an operator formulation of a periodic problem, and by the fixed point method, we prove the existence–uniqueness of a periodic solution of the 3D Kepler problem. In this manner, we show the stability of the He atom in the frame of classical electrodynamics. The same approach can be used for the investigation of a similar problem for atoms with more complicated structures. As a particular case, we obtain a confirmation of the hypotheses stated by Bohr and Sommerfeld (cf. [22,23]).

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Appendix A. Derivation of Initial Equations in Spherical Coordinates

We pass to the spherical coordinates, that is, the particle $P_n(n = 2, 3)$ is located at the point

$$\begin{array}{l}{x_1}^{(n)}(\theta) = \rho_n(\theta)\cos\varphi_n(\theta)\cos\lambda_n(\theta); \\ {x_2}^{(n)}(\theta) = \rho_n(\theta)\sin\varphi_n(\theta)\cos\lambda_n(\theta); \\ {x_3}^{(n)}(\theta) = \rho_n(\theta)\sin\ \lambda_n(\theta)\end{array}$$

where $\rho_n(\theta) \geq 0\,;\ \varphi_n(\theta) \geq 0\,;\ \lambda_n(\theta) \in \left[-\frac{\pi}{2} + \delta, \frac{\pi}{2} - \delta\right],\ 0 < \delta < \frac{\pi}{2}$, and then the velocities are

$$\begin{array}{l}{u_1}^{(n)} = \dot{\rho}_n\cos\varphi_n\cos\lambda_n - \rho_n\dot{\varphi}_n\sin\varphi_n\cos\lambda_n\ - \rho_n\dot{\lambda}_n\cos\varphi_n\sin\lambda_n \\ {u_2}^{(n)} = \dot{\rho}_n\sin\varphi_n\cos\lambda_n + \rho_n\dot{\varphi}_n\cos\varphi_n\ \cos\lambda_n - \rho_n\dot{\lambda}_n\sin\varphi_n\sin\lambda_n \\ {u_3}^{(n)} = \dot{\rho}_n\sin\lambda_n + \rho_n\ \dot{\lambda}_n\cos\lambda_n\end{array}$$

where $\dot{\rho}^{(2)} = d\rho^{(2)}/d\theta$. For the accelerations, we have

$$\begin{aligned}
\dot{u}_1{}^{(n)} &\approx \ddot{\rho}_n \cos\varphi_n \cos\lambda_n - \rho_n \ddot{\varphi}_n \sin\varphi_n \cos\lambda_n - \rho_n \ddot{\lambda}_n \cos\varphi_n \sin\lambda_n \\
\dot{u}_2{}^{(n)} &\approx \ddot{\rho}_n \sin\varphi_n \cos\lambda_n + \rho_n \ddot{\varphi}_n \cos\varphi_n \cos\lambda_n - \rho_n \ddot{\lambda}_n \sin\varphi_n \sin\lambda_n \\
\dot{u}_3{}^{(n)} &\approx \ddot{\rho}_n \sin\lambda_n + \rho_n \ddot{\lambda}_n \cos\lambda_n.
\end{aligned}$$

In the same manner, we obtain the expressions for the second derivatives $(n = 2, 3)$:

$$\begin{aligned}
\ddot{u}_1^{(n)} &\approx \dddot{\rho}_n \cos\varphi_n \cos\lambda_n - \dddot{\varphi}_n \rho_n \sin\varphi_n \cos\lambda_n - \dddot{\lambda}_n \rho_n \cos\varphi_n \sin\lambda_n \\
\ddot{u}_2^{(n)} &\approx \dddot{\rho}_n \sin\varphi_n \cos\lambda_n + \dddot{\varphi}_n \rho_n \cos\varphi_n \cos\lambda_n - \dddot{\lambda}_n \rho_n \sin\varphi_n \sin\lambda_n \\
\ddot{u}_3^{(n)} &\approx \dddot{\rho}_n \sin\lambda_n + \dddot{\lambda}_n \rho_n \cos\lambda_n
\end{aligned}$$

The condition (C) implies

$$\begin{aligned}
&\left\langle \vec{u}^{(2)}, \vec{u}^{(2)} \right\rangle = \dot{\rho}_2{}^2 + \rho_2{}^2 \dot{\varphi}_2{}^2 \cos^2\lambda_2 + \rho_2{}^2 \dot{\lambda}_2{}^2 \le \bar{c}^2 < c^2 \;\Rightarrow\; \frac{|\dot{\rho}_2|}{c} \approx 0,\; \frac{|\rho_2 \dot{\varphi}_2|}{c} \approx 0,\; \frac{|\rho_2 \dot{\lambda}_2|}{c} \approx 0; \\
&\left\langle \vec{u}^{(2)}, \vec{u}^{(2)} \right\rangle / c^2 \approx 0; \\
&\left\langle \dot{\vec{u}}^{(2)}, \dot{\vec{u}}^{(2)} \right\rangle = \ddot{\rho}_2{}^2 + \rho_2{}^2 \ddot{\varphi}_2{}^2 \cos^2\lambda_2 + \rho_2{}^2 \ddot{\lambda}_2{}^2; \; \left\langle \ddot{\vec{u}}^{(2)}, \ddot{\vec{u}}^{(2)} \right\rangle = \dddot{\rho}_2{}^2 + \rho_2{}^2 \dddot{\varphi}_2{}^2 \cos^2\lambda_2 + \rho_2{}^2 \dddot{\lambda}_2{}^2.
\end{aligned}$$

Further on, we have

$$\begin{aligned}
&\vec{\xi}^{(12)} = (\xi_1^{(12)}, \xi_1^{(12)}, \xi_1^{(12)}) = \left(x_1^{(1)}(t) - x_1^{(2)}(t - \tau_{12}(t)), x_2^{(1)}(t) - x_2^{(2)}(t - \tau_{12}(t)), x_3^{(1)}(t) - x_3^{(2)}(t - \tau_{12}(t)) \right) = \\
&= \left(-x_1^{(2)}(\theta), -x_2^{(2)}(\theta), -x_3^{(2)}(\theta) \right) = (-\rho_2 \cos\varphi_2 \cos\lambda_2, -\rho_2 \sin\varphi_2 \cos\lambda_2, -\rho_2 \sin\;\lambda_2); \\
&\vec{\xi}^{(13)} = (\xi_1^{(13)}, \xi_1^{(13)}, \xi_1^{(13)}) = \left(x_1^{(1)}(t) - x_1^{(3)}(t - \tau_{13}(t)), x_2^{(1)}(t) - x_2^{(3)}(t - \tau_{13}(t)), x_3^{(1)}(t) - x_3^{(3)}(t - \tau_{13}(t)) \right) = \\
&= \left(-x_1^{(3)}(\vartheta), -x_2^{(3)}(\vartheta), -x_3^{(3)}(\vartheta) \right) = (-\rho_3 \cos\varphi_3 \cos\lambda_3, -\rho_3 \sin\varphi_3 \cos\lambda_3, -\rho_3 \sin\;\lambda_3);
\end{aligned}$$

$$\begin{aligned}
\tau_{12}(t) &= \frac{1}{c} \sqrt{\sum_{\alpha=1}^{3} \left[x_\alpha^{(1)}(t) - x_\alpha^{(2)}(t - \tau_{12}(t) \right]^2} = \frac{1}{c} \sqrt{\sum_{\alpha=1}^{3} \left[-x_\alpha^{(2)}(t - \tau_{12}(t) \right]^2} = \frac{\rho_2(t - \tau_{12}(t))}{c} = \frac{\rho_2(\theta)}{c}; \\
\tau_{21} &= \frac{1}{c} \sqrt{\left\langle \vec{\xi}^{(21)}, \vec{\xi}^{(21)} \right\rangle} = \frac{1}{c} \sqrt{\left\langle \vec{x}^{(2)}, \vec{x}^{(2)} \right\rangle} = \frac{\rho_2}{c}; \\
\tau_{13}(t) &= \frac{1}{c} \sqrt{\sum_{\alpha=1}^{3} \left[x_\alpha^{(1)}(t) - x_\alpha^{(3)}(t - \tau_{13}(t) \right]^2} = \frac{1}{c} \sqrt{\sum_{\alpha=1}^{3} \left[-x_\alpha^{(2)}(t - \tau_{13}(t) \right]^2} = \frac{\rho_3(t - \tau_{13}(t))}{c} = \frac{\rho_3(\vartheta)}{c};
\end{aligned}$$

$$\begin{aligned}
&\xi^{(21)} = \left(\xi_1^{(21)}, \xi_2^{(21)}, \xi_3^{(21)}, \xi_4^{(21)} \right) = \left(\vec{\xi}^{(21)}, \xi_4^{(21)} \right) = \left(x_1^{(2)}(t) - x_1^{(1)}(t - \tau_{21}(t)), x_2^{(2)}(t) - x_2^{(1)}(t - \tau_{21}(t)), x_3^{(2)}(t) - x_3^{(1)}(t - \tau_{21}(t)), ic\tau_{21} \right) = \\
&= \left(x_1^{(2)}(t), x_2^{(2)}(t), x_3^{(2)}(t), ic\tau_{21} \right); \\
&\xi^{(23)} = \left(\xi_1^{(23)}, \xi_2^{(23)}, \xi_3^{(23)}, \xi_4^{(23)} \right) = \left(\vec{\xi}^{(23)}, \xi_4^{(23)} \right) = \left(x_1^{(2)}(t) - x_1^{(3)}(t - \tau_{23}), x_2^{(2)}(t) - x_2^{(3)}(t - \tau_{23}), x_3^{(2)}(t) - x_3^{(3)}(t - \tau_{23}), ic\tau_{23} \right) = \\
&= (\rho_2 \cos\varphi_2 \cos\lambda_2 - \rho_3 \cos\varphi_3 \cos\lambda_3,\; \rho_2 \sin\varphi_2 \cos\lambda_2 - \rho_3 \sin\varphi_3 \cos\lambda_3,\; \rho_2 \sin\lambda_2 - \rho_3 \sin\lambda_3); \\
&c^2 \tau_{23}{}^2 = (\rho_2 \cos\varphi_2 \cos\lambda_2 - \rho_3 \cos\varphi_3 \cos\lambda_3)^2 + (\rho_2 \sin\varphi_2 \cos\lambda_2 - \rho_3 \sin\varphi_3 \cos\lambda_3)^2 + (\rho_2 \sin\lambda_2 - \rho_3 \sin\lambda_3)^2 = \\
&= \rho_2{}^2 + \rho_3{}^2 - 2\rho_2\rho_3[\cos\lambda_2 \cos\lambda_3 \cos(\varphi_2 - \varphi_3) + \sin\lambda_2 \sin\lambda_3]; \\
&\xi^{(31)} = \left(\xi_1^{(31)}, \xi_2^{(31)}, \xi_3^{(31)}, \xi_4^{(31)} \right) = \left(\vec{\xi}^{(31)}, \xi_4^{(31)} \right) = \left(x_1^{(3)}(t) - x_1^{(1)}(t - \tau_{31}(t)), x_2^{(3)}(t) - x_2^{(1)}(t - \tau_{31}(t)), x_3^{(3)}(t) - x_3^{(1)}(t - \tau_{31}(t)), ic\tau_{31} \right) = \\
&= \left(x_1^{(3)}(t), x_2^{(3)}(t), x_3^{(3)}(t), ic\tau_{31} \right); \\
&\xi^{(32)} = \left(\xi_1^{(32)}, \xi_2^{(32)}, \xi_3^{(32)}, \xi_4^{(32)} \right) = \left(\vec{\xi}^{(32)}, \xi_4^{(32)} \right) = \left(x_1^{(3)}(t) - x_1^{(2)}(t - \tau_{32}), x_2^{(3)}(t) - x_2^{(2)}(t - \tau_{32}), x_3^{(3)}(t) - x_3^{(2)}(t - \tau_{32}), ic\tau_{32} \right) = \\
&= (\rho_3 \cos\varphi_3 \cos\lambda_3 - \rho_2 \cos\varphi_2 \cos\lambda_2,\; \rho_3 \sin\varphi_3 \cos\lambda_3 - \rho_2 \sin\varphi_2 \cos\lambda_2,\; \rho_3 \sin\lambda_3 - \rho_2 \sin\lambda_2); \\
&c^2 \tau_{32}{}^2 = (\rho_3 \cos\varphi_3 \cos\lambda_3 - \rho_2 \cos\varphi_2 \cos\lambda_2)^2 + (\rho_3 \sin\varphi_3 \cos\lambda_3 - \rho_2 \sin\varphi_2 \cos\lambda_2)^2 + (\rho_3 \sin\lambda_3 - \rho_2 \sin\lambda_2)^2 = \\
&= \rho_2{}^2 + \rho_3{}^2 - 2\rho_2\rho_3[\cos\lambda_2 \cos\lambda_3 \cos(\varphi_2 - \varphi_3) + \sin\lambda_2 \sin\lambda_3].
\end{aligned}$$

But

$$
\begin{aligned}
&\cos\lambda_2\cos\lambda_3\cos(\varphi_2-\varphi_3)+\sin\lambda_2\sin\lambda_3=\\
&=\tfrac{\cos(\lambda_2-\lambda_3)+\cos(\lambda_2+\lambda_3)}{2}\cos(\varphi_2-\varphi_3)+\tfrac{\cos(\lambda_2-\lambda_3)-\cos(\lambda_2+\lambda_3)}{2}=\\
&=\tfrac{\cos(\lambda_2-\lambda_3)[1+\cos(\varphi_2-\varphi_3)]-[\cos(\lambda_2+\lambda_3)-\cos(\lambda_2+\lambda_3)\cos(\varphi_2-\varphi_3)]}{2}=\\
&=\tfrac{\cos(\lambda_2-\lambda_3)2\cos^2\frac{\varphi_2-\varphi_3}{2}-\cos(\lambda_2+\lambda_3)2\sin^2\frac{\varphi_2-\varphi_3}{2}}{2}\le\cos^2\tfrac{\varphi_2-\varphi_3}{2}+\sin^2\tfrac{\varphi_2-\varphi_3}{2}=1.
\end{aligned}
$$

Therefore,

$$c^2{\tau_{23}}^2={\rho_2}^2+{\rho_3}^2-2\rho_2\rho_3[\cos\lambda_2\cos\lambda_3\cos(\varphi_2-\varphi_3)+\sin\lambda_2\sin\lambda_3]\ge(\rho_2-\rho_3)^2\Leftrightarrow\tau_{23}\ge\frac{|\rho_2(t)-\rho_3(t-\tau_{23})|}{c}$$

and $\tau_{32}\ge\frac{|\rho_3(t)-\rho_2(t-\tau_{32})|}{c}$.

For the accelerations, we obtain $(n=2,3)$

$$
\begin{aligned}
&\dot{u}_1{}^{(n)}=c\tfrac{\dot\rho_n\cos\varphi_n\cos\lambda_n-\rho_n\dot\varphi_n\sin\varphi_n\cos\lambda_n\ -\rho_n\dot\lambda_n\cos\varphi_n\sin\lambda_n-c\cos\varphi_n\cos\lambda_n}{\rho_n}\approx-\tfrac{c^2\cos\varphi_n\cos\lambda_n}{\rho_n}\equiv Q_1{}^{(n)},\\
&\dot{u}_2{}^{(n)}=c\tfrac{\dot\rho_n\sin\varphi_n\cos\lambda_n+\rho_n\dot\varphi_n\cos\varphi_n\cos\lambda_n-\rho_n\dot\lambda_n\sin\varphi_n\sin\lambda_n-c\sin\varphi_n\cos\lambda_n}{\rho_n}\approx-\tfrac{c^2\sin\varphi_n\cos\lambda_n}{\rho_n}\equiv Q_2{}^{(n)},\\
&\dot{u}_3{}^{(n)}=c\tfrac{\dot\rho_n\sin\lambda_n+\rho_n\ \dot\lambda_n\cos\lambda_n-c\sin\ \lambda_n}{\rho_n}\approx-\tfrac{c^2\sin\lambda_n}{\rho_n}\equiv Q_3{}^{(n)}\ ,\ \ (n=2,3).
\end{aligned}
$$

Passing to the spherical coordinates, we obtain

$$
\begin{aligned}
&\ddot\rho_n\cos\varphi_n\cos\lambda_n-\rho_n\ddot\varphi_n\sin\varphi_n\cos\lambda_n-\rho_n\ddot\lambda_n\cos\varphi_n\sin\lambda_n=\\
&=2\dot\rho_n\dot\varphi_n\sin\varphi_n\cos\lambda_n+2\dot\rho_n\dot\lambda_n\cos\varphi_n\sin\lambda_n-2\rho_n\dot\varphi_n\lambda_n\sin\varphi_n\sin\lambda_n+\rho_n{\dot\varphi_n}^2\cos\varphi_n\cos\lambda_n+\rho_n{\dot\lambda_n}^2\cos\varphi_n\cos\lambda_n-\tfrac{c^2\cos\varphi_n\cos\lambda_n}{\rho_n};\\
&\ddot\rho_n\sin\varphi_n\cos\lambda_n+\rho_n\ddot\varphi_n\cos\varphi_n\cos\lambda_n-\rho_n\ddot\lambda_n\sin\varphi_n\sin\lambda_n=\\
&=-2\dot\rho_n\dot\varphi_n\cos\varphi_n\cos\lambda_n+2\dot\rho_n\dot\lambda_n\sin\varphi_n\sin\lambda_n+2\rho_n\dot\varphi_n\dot\lambda_n\cos\varphi_n\ \sin\lambda_n+\rho_n{\dot\varphi_n}^2\sin\varphi_n\ \cos\lambda_n+\rho_n{\dot\lambda_n}^2\sin\varphi_n\cos\lambda_n-\tfrac{c^2\sin\varphi_n\cos\lambda_n}{\rho_n};\\
&\ddot\rho_n\sin\lambda_n+\rho\ _n\ddot\lambda_n\cos\lambda_n=-2\dot\rho_n\dot\lambda_n\cos\lambda_n+\rho_n\ {\dot\lambda_n}^2\sin\lambda_n-\tfrac{c^2\sin\ \lambda_n}{\rho_n}.
\end{aligned}
\tag{A1}
$$

We introduce denotations:

$$
\begin{aligned}
&P_1{}^{(n)}=2\dot\rho_n\dot\varphi_n\sin\varphi_n\cos\lambda_n+2\dot\rho_n\dot\lambda_n\cos\varphi_n\sin\lambda_n-2\rho_n\dot\varphi_n\dot\lambda_n\sin\varphi_n\sin\lambda_n+\rho_n{\dot\varphi_n}^2\cos\varphi_n\cos\lambda_n+\rho_n{\dot\lambda_n}^2\cos\varphi_n\cos\lambda_n;\\
&P_2{}^{(n)}=-2\dot\rho_n\dot\varphi_n\cos\varphi_n\cos\lambda_n+2\dot\rho_n\dot\lambda_n\sin\varphi_n\sin\lambda_n+2\rho_n\dot\varphi_n\dot\lambda_n\cos\varphi_n\ \sin\lambda_n+\rho_n{\dot\varphi_n}^2\sin\varphi_n\ \cos\lambda_n+\rho_n{\dot\lambda_n}^2\sin\varphi_n\cos\lambda_n;\\
&P_3{}^{(n)}=-2\dot\rho_n\dot\lambda_n\cos\lambda_n+\rho_n\ {\dot\lambda_n}^2\sin\lambda_n.
\end{aligned}
$$

Then, (A1) takes the following form:

$$
\begin{aligned}
&\ddot\rho_n\cos\varphi_n\cos\lambda_n-\ddot\varphi_n\rho_n\sin\varphi_n\cos\lambda_n-\ddot\lambda_n\rho_n\cos\varphi_n\sin\lambda_n=P_1{}^{(n)}+Q_1{}^{(n)}\\
&\ddot\rho_n\sin\varphi_n\cos\lambda_n+\ddot\varphi_n\rho_n\cos\varphi_n\ \cos\lambda_n-\ddot\lambda_n\rho_n\sin\varphi_n\sin\lambda_n=P_2{}^{(n)}+Q_2{}^{(n)};\\
&\ddot\rho_n\sin\lambda_n+\ \ddot\lambda_n\rho_n\cos\lambda_n=P_3{}^{(n)}+Q_3{}^{(n)}.
\end{aligned}
$$

To solve the last system with respect to $\ddot\rho_n$, $\ddot\varphi_n$, $\ddot\lambda_n$, we notice that

$$\begin{vmatrix}\cos\varphi_n\cos\lambda_n & -\rho_n\sin\varphi_n\cos\lambda_n & -\rho_n\cos\varphi_n\sin\lambda_n\\ \sin\varphi_n\cos\lambda_n & \rho_n\cos\varphi_n\ \cos\lambda_n & -\rho_n\sin\varphi_n\sin\lambda_n\\ \sin\lambda_n & 0 & \rho_n\cos\lambda_n\end{vmatrix}={\rho_n}^2\cos\lambda_n>0$$

and then, we obtain the initial equations $(n=2,3)$:

$$
\begin{aligned}
&\ddot\rho_n=\rho_n{\dot\varphi_n}^2\cos^2\lambda_n+\rho_n{\dot\lambda_n}^2-\tfrac{c^2}{\rho_n},\\
&\ddot\varphi_n=2\dot\varphi_n\dot\lambda_n tg\lambda_n-\tfrac{2\dot\rho_n\dot\varphi_n}{\rho_n},\\
&\ddot\lambda_n=-\tfrac{2\dot\rho_n\dot\lambda_n+\rho_n{\dot\varphi_n}^2\sin\lambda_n\ \cos\lambda_n}{\rho_n}.
\end{aligned}
$$

Appendix B. Final Form of Basic Equations

First, we calculate $\left\langle \dot{\vec{u}}^{(n)}, \dot{\vec{u}}^{(n)} \right\rangle = \ddot{\rho}_n{}^2 + \rho_n{}^2 \ddot{\varphi}_n{}^2 \cos^2 \lambda_n + \rho_n{}^2 \ddot{\lambda}_n{}^2$;

$$\left\langle \vec{\xi}^{(23)}, \dot{\vec{u}}^{(3)} \right\rangle = [\rho_2 \cos(\varphi_2 - \varphi_3) \cos \lambda_2 \cos \lambda_3 + \rho_2 \sin \lambda_2 \sin \lambda_3 - \rho_3]\ddot{\rho}_3 +$$
$$+\rho_2\rho_3 \sin(\varphi_2 - \varphi_3) \cos \lambda_2 \cos \lambda_3 \ddot{\varphi}_3 + \rho_2\rho_3[\sin \lambda_2 \cos \lambda_3 - \cos(\varphi_2 - \varphi_3) \sin \lambda_3 \cos \lambda_2]\ddot{\lambda}_3;$$
$$\left\langle \vec{\xi}^{(32)}, \dot{\vec{u}}^{(2)} \right\rangle = [\rho_3 \cos(\varphi_3 - \varphi_2) \cos \lambda_3 \cos \lambda_2 + \rho_3 \sin \lambda_3 \sin \lambda_2 - \rho_2]\ddot{\rho}_2 +$$
$$+ \sin(\varphi_3 - \varphi_2) \cos \lambda_3 \cos \lambda_2 \rho_2\rho_3 \ddot{\varphi}_2 + [\sin \lambda_3 \cos \lambda_2) - \cos(\varphi_3 - \varphi_2) \sin \lambda_2 \cos \lambda_3]\rho_2\rho_3\ddot{\lambda}_2.$$

Introduce denotations

$$P_\alpha^{(2)} = G_\alpha^{(21)} + G_\alpha^{(23)} + G_\alpha^{(2)rad}; \quad P_\alpha^{(3)} = G_\alpha^{(31)} + G_\alpha^{(32)} + G_\alpha^{(3)rad}; \ (\alpha = 1, 2, 3).$$

Then, the system of equations of motion becomes

$$\begin{aligned}
\dot{u}_1^{(n)} &= \frac{c^2 - \left(u_1^{(n)}\right)^2}{c^2} P_1^{(n)} - \frac{u_1^{(n)} u_2^{(n)}}{c^2} P_2^{(n)} - \frac{u_1^{(n)} u_3^{(n)}}{c^2} P_3^{(n)} \equiv U_1^{(n)};\\
\dot{u}_2^{(n)} &= -\frac{u_1^{(n)} u_2^{(n)}}{c^2} P_1^{(n)} + \frac{c^2 - (u_2^{(n)})^2}{c^2} P_2^{(n)} - \frac{u_2^{(n)} u_3^{(n)}}{c^2} P_3^{(n)} \equiv U_2^{(n)};\\
\dot{u}_3^{(n)} &= -\frac{u_1^{(n)} u_3^{(n)}}{c^2} P_1^{(n)} - \frac{u_2^{(n)} u_3^{(n)}}{c^2} P_2^{(n)} + \frac{c^2 - (u_3^{(n)})^2}{c^2} P_3^{(n)} \equiv U_3^{(n)}, (n = 2, 3).
\end{aligned}$$

But

$$\begin{aligned}
\dot{u}_1{}^{(n)} &\approx \ddot{\rho}_n \cos \varphi_n \cos \lambda_n - \rho_n \ddot{\varphi}_n \sin \varphi_n \cos \lambda_n - \rho_n \ddot{\lambda}_n \cos \varphi_n \sin \lambda_n\\
\dot{u}_2{}^{(n)} &\approx \ddot{\rho}_n \sin \varphi_n \cos \lambda_n + \rho_n \ddot{\varphi}_n \cos \varphi_n \cos \lambda_n - \rho_n \ddot{\lambda}_n \sin \varphi_n \sin \lambda_n\\
\dot{u}_3{}^{(n)} &\approx \ddot{\rho}_n \sin \lambda_n + \rho_n \ddot{\lambda}_n \cos \lambda_n
\end{aligned}$$

and since $\left\langle \vec{u}^{(n)}, \vec{u}^{(n)} \right\rangle / c^2 \approx 0$, we infer

$$\begin{aligned}
U_1^{(n)} &= \frac{c^2 - \left(u_1^{(n)}\right)^2}{c^2} P_1^{(n)} - \frac{u_1^{(n)} u_2^{(n)}}{c^2} P_2^{(n)} - \frac{u_1^{(n)} u_3^{(n)}}{c^2} P_3^{(n)} \approx P_1^{(n)};\\
U_2^{(n)} &= -\frac{u_1^{(n)} u_2^{(n)}}{c^2} P_1^{(n)} + \frac{c^2 - (u_2^{(n)})^2}{c^2} P_2^{(n)} - \frac{u_2^{(n)} u_3^{(n)}}{c^2} P_3^{(n)} \approx P_2^{(n)};\\
U_3^{(n)} &= -\frac{u_1^{(n)} u_3^{(n)}}{c^2} P_1^{(n)} - \frac{u_2^{(n)} u_3^{(n)}}{c^2} P_2^{(n)} + \frac{c^2 - (u_3^{(n)})^2}{c^2} P_3^{(n)} \approx P_3^{(n)}, (n = 2, 3)
\end{aligned}$$

and therefore,

$$\begin{aligned}
&\ddot{\rho}_n \cos \varphi_n \cos \lambda_n - \rho_n \ddot{\varphi}_n \sin \varphi_n \cos \lambda_n - \rho_n \ddot{\lambda}_n \cos \varphi_n \sin \lambda_n = P_1^{(n)}\\
&\ddot{\rho}_n \sin \varphi_n \cos \lambda_n + \rho_n \ddot{\varphi}_n \cos \varphi_n \cos \lambda_n - \rho_n \ddot{\lambda}_n \sin \varphi_n \sin \lambda_n = P_2^{(n)}\\
&\ddot{\rho}_n \sin \lambda_n + \rho_n \ddot{\lambda}_n \cos \lambda_n = P_3^{(n)}.
\end{aligned}$$

Solving the above system with respect to $\ddot{\rho}_n$, $\ddot{\varphi}_n$, $\ddot{\lambda}_n$, we obtain

$$\begin{aligned}
\ddot{\rho}_n &= P_1^{(n)} \cos \varphi_n \cos \lambda_n + P_2^{(n)} \sin \varphi_n \cos \lambda_n + P_3^{(n)} \sin \lambda_n;\\
\ddot{\varphi}_n &= \frac{-P_1^{(n)} \sin \varphi_n + P_2^{(n)} \cos \varphi_n}{\rho_n \cos \lambda_n};\\
\ddot{\lambda}_n &= \frac{-P_1^{(n)} \cos \varphi_n \sin \lambda_n - P_2^{(n)} \sin \varphi_n \sin \lambda_n + P_3^{(n)} \cos \lambda_n}{\rho_n} (n = 2, 3)
\end{aligned} \quad . \tag{A2}$$

Recall that $\varphi_{23} = \varphi_2(t) - \varphi_3(t - \tau_{23})$ and $\varphi_{32} = \varphi_2(t - \tau_{32}) - \varphi_3(t)$, and, in view of

$$G_\alpha^{(23)} \approx \frac{e_2 e_3}{m_2 c^3}\left(\frac{\xi_\alpha^{(23)}}{{\tau_{23}}^3}\frac{\left\langle \vec{\xi}^{(23)}, \dot{\vec{u}}^{(3)}\right\rangle}{c^3} - \frac{\dot{u}_\alpha^{(3)}}{c\tau_{23}}\right),\ G_\alpha^{(32)} \approx \frac{e_2 e_3}{m_3 c^3}\left(\frac{\xi_\alpha^{(32)}}{{\tau_{32}}^3}\frac{\left\langle \vec{\xi}^{(32)}, \dot{\vec{u}}^{(3)}\right\rangle}{c^3} - \frac{\dot{u}_\alpha^{(2)}}{c\tau_{32}}\right);$$

$$P_\alpha^{(2)} = G_\alpha^{(21)} + G_\alpha^{(23)} + G_\alpha^{(2)rad} \approx \frac{e_2 e_1}{m_2 c^4}\frac{\xi_\alpha^{(21)}}{{\tau_{21}}^3} + \frac{e_2 e_3}{m_2 c^3}\left(\frac{\xi_\alpha^{(23)}}{{\tau_{23}}^3}\frac{\left\langle \vec{\xi}^{(23)}, \dot{\vec{u}}^{(3)}\right\rangle}{c^3} - \frac{\dot{u}_\alpha^{(3)}}{c\tau_{23}}\right) - \frac{e_2^2}{m_2 c^3}\ddot{u}_\alpha^{(2)};$$

$$P_\alpha^{(3)} = G_\alpha^{(31)} + G_\alpha^{(32)} + G_\alpha^{(3)rad} \approx \frac{e_3 e_1}{m_3}\frac{\xi_\alpha^{(31)}}{c^4{\tau_{31}}^3} + \frac{e_2 e_3}{m_3 c^3}\left(\frac{\xi_\alpha^{(32)}}{{\tau_{32}}^3}\frac{\left\langle \vec{\xi}^{(32)}, \dot{\vec{u}}^{(2)}\right\rangle}{c^3} - \frac{\dot{u}_\alpha^{(2)}}{c\tau_{32}}\right) - \frac{e_3^2}{m_3 c^3}\ddot{u}_\alpha^{(3)}$$

$$(\alpha = 1, 2, 3)$$

we transform the right-hand sides of the equations of (A2).

Firstly,

$$\begin{aligned}
\ddot{\rho}_2 &= P_1^{(2)}\cos\varphi_2\cos\lambda_2 + P_2^{(2)}\sin\varphi_2\cos\lambda_2 + P_3^{(2)}\sin\lambda_2 = \\
&= \cos\varphi_2\cos\lambda_2\left[\frac{e_2 e_1}{m_2 c^4}\frac{\xi_1^{(21)}}{{\tau_{21}}^3} + \frac{e_2 e_3}{m_2 c^3}\left(\frac{\xi_1^{(23)}}{{\tau_{23}}^3}\frac{\left\langle \vec{\xi}^{(23)}, \dot{\vec{u}}^{(3)}\right\rangle}{c^3} - \frac{\dot{u}_1^{(3)}}{c\tau_{23}}\right) - \frac{e_2^2}{m_2 c^3}\ddot{u}_1^{(2)}\right] + \\
&+ \sin\varphi_2\cos\lambda_2\left[\frac{e_2 e_1}{m_2 c^4}\frac{\xi_2^{(21)}}{{\tau_{21}}^3} + \frac{e_2 e_3}{m_2 c^3}\left(\frac{\xi_2^{(23)}}{{\tau_{23}}^3}\frac{\left\langle \vec{\xi}^{(23)}, \dot{\vec{u}}^{(3)}\right\rangle}{c^3} - \frac{\dot{u}_2^{(3)}}{c\tau_{23}}\right) - \frac{e_2^2}{m_2 c^3}\ddot{u}_2^{(2)}\right] + \\
&+ \sin\lambda_2\left[\frac{e_2 e_1}{m_2 c^4}\frac{\xi_3^{(21)}}{{\tau_{21}}^3} + \frac{e_2 e_3}{m_2 c^3}\left(\frac{\xi_3^{(23)}}{{\tau_{23}}^3}\frac{\left\langle \vec{\xi}^{(23)}, \dot{\vec{u}}^{(3)}\right\rangle}{c^3} - \frac{\dot{u}_3^{(3)}}{c\tau_{23}}\right) - \frac{e_2^2}{m_2 c^3}\ddot{u}_3^{(2)}\right] =
\end{aligned}$$

$$\begin{aligned}
&= \frac{e_2 e_1}{m_2 c^4}\frac{\xi_1^{(21)}\cos\varphi_2\cos\lambda_2 + \xi_2^{(21)}\sin\varphi_2\cos\lambda_2 + \xi_3^{(21)}\sin\lambda_2}{{\tau_{21}}^3} + \frac{e_2 e_3}{m_2 c^4}\left(\xi_1^{(23)}\cos\varphi_2\cos\lambda_2 + \xi_2^{(23)}\sin\varphi_2\cos\lambda_2 + \xi_3^{(23)}\sin\lambda_2\right)\frac{\left\langle \vec{\xi}^{(23)}, \dot{\vec{u}}^{(3)}\right\rangle}{c^2{\tau_{23}}^3} - \\
&- \frac{e_2 e_3}{m_2 c^3 \tau_{23}}\frac{\dot{u}_1^{(3)}\cos\varphi_2\cos\lambda_2 + \dot{u}_2^{(3)}\sin\varphi_2\cos\lambda_2 + \dot{u}_3^{(3)}\sin\lambda_2}{c} - \frac{e_2^2}{m_2 c^3}\left(\ddot{u}_1^{(2)}\cos\varphi_2\cos\lambda_2 + \ddot{u}_2^{(2)}\sin\varphi_2\cos\lambda_2 + \ddot{u}_3^{(2)}\sin\lambda_2\right) \approx \\
&\approx \frac{e_2 e_1}{m_2 c}\frac{1}{{\rho_2}^2} - \frac{e_2 e_3}{m_2}\frac{\rho_2 - \rho_3(\cos\varphi_{23}\cos\lambda_2\cos\lambda_3 + \sin\lambda_2\sin\lambda_3)}{c^3}\frac{\left\langle \vec{\xi}^{(23)}, \dot{\vec{u}}^{(3)}\right\rangle}{c^3{\tau_{23}}^3} - \\
&- \frac{e_2 e_3}{m_2}\frac{\ddot{\rho}_3(\cos\varphi_{23}\cos\lambda_2\cos\lambda_3 + \sin\lambda_2\sin\lambda_3) + \rho_3\ddot{\varphi}_3\sin\varphi_{23}\cos\lambda_2\cos\lambda_3 + \rho_3\ddot{\lambda}_3(\sin\lambda_2\cos\lambda_3 - \cos\varphi_{23}\sin\lambda_{.3}\cos\lambda_2)}{c^4\tau_{23}} - \frac{e_2^2\,\dddot{\rho}_2}{m_2 c^3}.
\end{aligned}$$

For the second equation of (A2), we have

$$\begin{aligned}
&P_\alpha^{(2)} = G_\alpha^{(21)} + G_\alpha^{(23)} + G_\alpha^{(2)rad} \approx \frac{e_2 e_1}{m_2 c^4}\frac{x_\alpha^{(2)}}{{\tau_{21}}^3} + \frac{e_2 e_3}{m_2 c^3}\left(\frac{\xi_\alpha^{(23)}}{{\tau_{23}}^3}\frac{\left\langle \vec{\xi}^{(23)}, \dot{\vec{u}}^{(3)}\right\rangle}{c^3} - \frac{\dot{u}_\alpha^{(3)}}{c\tau_{23}}\right) - \frac{e_2^2}{m_2 c^3}\ddot{u}_\alpha^{(2)}\ ; \\
&\ddot{\varphi}_2 = \frac{-\sin\varphi_2 P_1^{(2)} + \cos\varphi_2 P_2^{(2)}}{\rho_2\cos\lambda_2} = \frac{1}{\rho_2\cos\lambda_2}\left[\frac{e_2 e_1}{m_2 c^4}\frac{-x_1^{(2)}\sin\varphi_2 + x_2^{(2)}\cos\varphi_2}{{\tau_{21}}^3} + \right. \\
&\left. + \frac{e_2 e_3}{m_2 c^4}\left(\left(-\xi_1^{(23)}\sin\varphi_2 + \xi_2^{(23)}\cos\varphi_2\right)\frac{\left\langle \vec{\xi}^{(23)}, \dot{\vec{u}}^{(3)}\right\rangle}{c^2{\tau_{23}}^3} + \frac{\dot{u}_1^{(3)}\sin\varphi_2 - \dot{u}_2^{(3)}\cos\varphi_2}{\tau_{23}}\right) - \frac{e_2^2}{m_2 c^3}\left(\ddot{u}_\alpha^{(2)}\sin\varphi_2 - \ddot{u}_2^{(2)}\cos\varphi_2\right)\right] = \\
&\quad = \frac{1}{\rho_2\cos\lambda_2}\frac{e_2 e_3}{m_2 c^3}\left[\frac{\rho_3\sin\varphi_{23}\cos\lambda_3}{{\tau_{23}}^3}\frac{\left\langle \vec{\xi}^{(23)}, \dot{\vec{u}}^{(3)}\right\rangle}{c^3} + \frac{\ddot{\rho}_3\cos\lambda_3\sin\varphi_{23} + \rho_3\ddot{\varphi}_3\cos_3\cos\varphi_{23} - \rho_3\ddot{\lambda}_3\sin\lambda_3\sin\varphi_{23}}{c\tau_{23}} - \dddot{\varphi}_2\rho_2\cos\lambda_2\right].
\end{aligned} \tag{A3}$$

For the third equation of (A2), we have

$$\ddot{\lambda}_2 = \frac{-\cos\varphi_2 \sin\lambda_2}{\rho_2} P_1^{(2)} + \frac{-\sin\varphi_2 \sin\lambda_2}{\rho_2} P_2^{(2)} + \frac{\cos\lambda_2}{\rho_2} P_3^{(2)} =$$

$$= \frac{-\cos\varphi_2 \sin\lambda_2}{\rho_2} \left(\frac{e_2 e_1}{m_2 c^4} \frac{x_1^{(2)}}{\tau_{21}{}^3} + \frac{e_2 e_3}{m_2 c^3} \left(\frac{\xi_1^{(23)}}{\tau_{23}{}^3} \frac{\left\langle \vec{\xi}^{(23)}, \dot{\vec{u}}^{(3)} \right\rangle}{c^3} - \frac{\dot{u}_1^{(3)}}{c\tau_{23}} \right) - \frac{e_2^2}{m_2 c^3} \ddot{u}_1^{(2)} \right) +$$

$$+ \frac{-\sin\varphi_2 \sin\lambda_2}{\rho_2} \left(\frac{e_2 e_1}{m_2 c^4} \frac{x_2^{(2)}}{\tau_{21}{}^3} + \frac{e_2 e_3}{m_2 c^3} \left(\frac{\xi_2^{(23)}}{\tau_{23}{}^3} \frac{\left\langle \vec{\xi}^{(23)}, \dot{\vec{u}}^{(3)} \right\rangle}{c^3} - \frac{\dot{u}_2^{(3)}}{c\tau_{23}} \right) - \frac{e_2^2}{m_2 c^3} \ddot{u}_2^{(2)} \right) +$$

$$+ \frac{\cos\lambda_2}{\rho_2} \left(\frac{e_2 e_1}{m_2 c^4} \frac{x_3^{(2)}}{\tau_{21}{}^3} + \frac{e_2 e_3}{m_2 c^3} \left(\frac{\xi_3^{(23)}}{\tau_{23}{}^3} \frac{\left\langle \vec{\xi}^{(23)}, \dot{\vec{u}}^{(3)} \right\rangle}{c^3} - \frac{\dot{u}_3^{(3)}}{c\tau_{23}} \right) - \frac{e_2^2}{m_2 c^3} \ddot{u}_3^{(2)} \right) =$$

$$= \frac{e_2 e_3}{m_2 c^3} \frac{\rho_3 \cos\varphi_{23} \sin\lambda_2 \cos\lambda_3}{\tau_{23}{}^3} \frac{\left\langle \vec{\xi}^{(23)}, \dot{\vec{u}}^{(3)} \right\rangle}{c^3 \rho_2} +$$

$$+ \frac{e_2 e_3}{m_2 c^3} \frac{\ddot{\rho}_3(\cos\varphi_{23}\cos\lambda_3\sin\lambda_2 - \sin\lambda_3\cos\lambda_2) + \rho_3\ddot{\varphi}_3 \sin\varphi_{23}\sin\lambda_2\cos\lambda_3 - \rho_3\ddot{\lambda}_3(\cos\varphi_{23}\sin\lambda_2\sin\lambda_3 + \cos\lambda_2\cos\lambda_3)}{c\tau_{23}\rho_2} - \frac{e_2^2}{m_2 c^3} \dddot{\lambda}_2.$$

Appendix C

Since, in the Main Theorem, $\left|r_0{}^{(m)}\right| (m = 1, 2, \ldots)$ should be sufficiently small, we take $\left|r_0{}^{(m)}\right| = 1/\mu (m = 1, 2, \ldots)$, then,

$$|r_n(t)| = \left| r_{n0} + \int_{T_k}^{t} r_n{}^{(1)}(t_1) dt_1 \right| = \left| r_{n0} + \int_{T_k}^{t} \left(r_{n0}{}^{(1)} + \int_{T_k}^{t_1} r_n{}^{(2)}(t_2) dt_2 \right) dt_1 \right| = \ldots =$$

$$= \left| \int_{T_k}^{t} r_{n0}{}^{(1)} dt_1 + \int_{T_k}^{t}\int_{T_k}^{t_1} r_{n0}{}^{(2)} dt_2 dt_1 + \ldots + \int_{T_k}^{t}\int_{T_k}^{t_1} \ldots \int_{T_k}^{t_{m-1}} r_n{}^{(m)}(t_m) dt_m \ldots dt_1 \right| \le$$

$$\frac{\left| r_{n0}{}^{(1)} \right|}{\mu} e^{\mu(t-T_k)} + \frac{\left| r_{n0}{}^{(2)} \right|}{\mu^2} e^{\mu(t-T_k)} + \ldots + \frac{\omega^m}{\mu^m} R_n e^{\mu(t-T_k)} \le$$

$$\le \frac{1}{\mu^2} e^{\mu T} + \frac{1}{\mu^3} e^{\mu T} + \ldots + \frac{\omega^m}{\mu^m} R_n e^{\mu(t-T_k)} \approx \frac{\omega^m}{\mu^m} R_n e^{\mu(t-T_k)},$$

That means $|r_n(t)| \le \frac{\omega^m}{\mu^m} R_n e^{\mu(t-T_k)} (n = 2, 3)$, because $\frac{1}{\mu} \approx 0$.

In the same way, $\left| r_n{}^{(1)}(t) \right| \le \frac{\omega^m}{\mu^m} \omega R_n e^{\mu(t-T_k)}$, $\left| r_n{}^{(2)}(t) \right| \le \frac{\omega^m}{\mu^m} \omega^2 R_n e^{\mu(t-T_k)}$, and so on.

Appendix D. Estimates for the Higher Derivatives

Assuming $\beta \approx 0$, we obtain

$$\frac{d\tau_{pq}(t)}{dt} = \frac{\left\langle \vec{\xi}^{(pq)}, \vec{u}^{(p)} \right\rangle - \left\langle \vec{\xi}^{(pq)}, \vec{u}^{(q)} \right\rangle}{c\sqrt{\left\langle \vec{\xi}^{(pq)}, \vec{\xi}^{(pq)} \right\rangle} - \left\langle \vec{\xi}^{(pq)}, \vec{u}^{(q)} \right\rangle} = \frac{\left\langle \vec{\xi}^{(pq)}, \vec{u}^{(p)} - \vec{u}^{(q)} \right\rangle}{c^2 \tau_{pq} - \left\langle \vec{\xi}^{(pq)}, \vec{u}^{(q)} \right\rangle} \approx \frac{2\beta}{1-\beta} < 1 \Leftrightarrow \beta = \frac{1}{137} < \frac{1}{3} < 1$$

$$\left| \frac{d\xi_2^{(23)}}{dt} \right| = \left| \frac{d\left[x_2^{(2)}(t) - x_2^{(3)}(t - \tau_{23}) \right]}{dt} \right| = \left| u_2^{(2)}(t) - u_2^{(3)}(t - \tau_{23})(1 - \dot{\tau}_{23}) \right| \le 2\bar{c}.$$

Then,

$$\left\langle \vec{u}^{(n)}, \vec{u}^{(n)} \right\rangle = \dot{\rho}_n{}^2 + \rho_n{}^2 \dot{\varphi}_n{}^2 \cos^2\lambda_n + \rho_n{}^2 \dot{\lambda}_n{}^2 \le \bar{c}^2 < c^2 ; \left\langle \vec{u}^{(2)}, \vec{u}^{(2)} \right\rangle / c^2 \approx 0;$$

$$\sqrt{\left\langle \dot{\vec{u}}^{(n)}, \dot{\vec{u}}^{(n)} \right\rangle} = \sqrt{\dot{r}_n{}^2 + \rho_n{}^2 \dot{\phi}_n{}^2 \cos^2\lambda_n + \rho_n{}^2 \dot{\eta}_n{}^2} \le \sqrt{\omega_n{}^2 (R_n{}^2 + \rho_n{}^2 \Phi_n{}^2 + \rho_n{}^2 \Upsilon_n{}^2) e^{2\mu(t-T_k)}} \le \omega_n \bar{c} e^{\mu(t-T_k)};$$

$$\sqrt{\left\langle \ddot{\vec{u}}^{(n)}, \ddot{\vec{u}}^{(n)} \right\rangle} = \sqrt{\ddot{r}_n{}^2 + \rho_n{}^2 \ddot{\underline{\phi}}_n{}^2 \cos^2\lambda_n + \rho_n{}^2 \ddot{\eta}_n{}^2} \le \omega_n{}^2 \bar{c} e^{\mu(t-T_k)};$$

$$\sqrt{\left\langle \dddot{\vec{u}}^{(n)}, \dddot{\vec{u}}^{(n)} \right\rangle} \le \sqrt{\dddot{r}_n{}^2 + \rho_n{}^2 \dddot{\phi}_n{}^2 \cos^2\lambda_n + \rho_n{}^2 \dddot{\eta}_n{}^2} \le \omega_n{}^3 \bar{c} e^{\mu(t-T_k)}, \frac{1}{\tau_{23}} \le \frac{c}{\Delta}; \frac{1}{\tau_{32}} \le \frac{c}{\Delta}.$$

Furthermore,

$$\frac{d^2B_{r_2}^{(k)}}{dt^2}=\frac{dG_{r_2}}{dt}=\frac{d\left(P_1^{(2)}\cos\varphi_2\cos\lambda_2+P_2^{(2)}\sin\varphi_2\cos\lambda_2+P_3^{(2)}\sin\lambda_2\right)}{dt}=$$
$$=\dot{P}_1^{(2)}\cos\varphi_2\cos\lambda_2-P_1^{(2)}\phi_2\sin\varphi_2\cos\lambda_2-P_1^{(2)}\eta_2\cos\varphi_2\sin\lambda_2+$$
$$+\dot{P}_2^{(2)}\sin\varphi_2\cos\lambda_2+P_2^{(2)}\phi_2\cos\varphi_2\cos\lambda_2-P_2^{(2)}\eta_2\sin\varphi_2\sin\lambda_2+\dot{P}_3^{(2)}\sin\lambda_2+P_3^{(2)}\eta_2\cos\lambda_2=$$
$$=\frac{d}{dt}\left[\frac{e_2e_1}{m_2c^4}\frac{x_1^{(2)}}{{\tau_{21}}^3}+\frac{e_2e_3}{m_2c^3}\left(\frac{\xi_1^{(23)}}{{\tau_{23}}^3}\frac{\left\langle\vec{\xi}^{(23)},\dot{\vec{u}}^{(3)}\right\rangle}{c^3}-\frac{\dot{u}_1^{(3)}}{c\tau_{23}}\right)-\frac{e_2^2}{m_2c^3}\ddot{u}_1^{(2)}\right]\cos\varphi_2\cos\lambda_2-$$
$$-\left[\frac{e_2e_1}{m_2c^4}\frac{x_1^{(2)}}{{\tau_{21}}^3}+\frac{e_2e_3}{m_2c^3}\left(\frac{\xi_1^{(23)}}{{\tau_{23}}^3}\frac{\left\langle\vec{\xi}^{(23)},\dot{\vec{u}}^{(3)}\right\rangle}{c^3}-\frac{\dot{u}_1^{(3)}}{c\tau_{23}}\right)-\frac{e_2^2}{m_2c^3}\ddot{u}_1^{(2)}\right](\phi_2\sin\varphi_2\cos\lambda_2+\eta_2\cos\varphi_2\sin\lambda_2)+$$
$$+\frac{d}{dt}\left[\frac{e_2e_1}{m_2c^4}\frac{x_2^{(2)}}{{\tau_{21}}^3}+\frac{e_2e_3}{m_2c^3}\left(\frac{\xi_2^{(23)}}{{\tau_{23}}^3}\frac{\left\langle\vec{\xi}^{(23)},\dot{\vec{u}}^{(3)}\right\rangle}{c^3}-\frac{\dot{u}_2^{(3)}}{c\tau_{23}}\right)-\frac{e_2^2}{m_2c^3}\ddot{u}_2^{(2)}\right]\sin\varphi_2\cos\lambda_2+$$
$$+\left[\frac{e_2e_1}{m_2c^4}\frac{x_2^{(2)}}{{\tau_{21}}^3}+\frac{e_2e_3}{m_2c^3}\left(\frac{\xi_2^{(23)}}{{\tau_{23}}^3}\frac{\left\langle\vec{\xi}^{(23)},\dot{\vec{u}}^{(3)}\right\rangle}{c^3}-\frac{\dot{u}_2^{(3)}}{c\tau_{23}}\right)-\frac{e_2^2}{m_2c^3}\ddot{u}_2^{(2)}\right](\phi_2\cos\varphi_2\cos\lambda_2-\eta_2\sin\varphi_2\sin\lambda_2)+$$
$$+\frac{d}{dt}\left[\frac{e_2e_1}{m_2c^4}\frac{x_3^{(2)}}{{\tau_{21}}^3}+\frac{e_2e_3}{m_2c^3}\left(\frac{\xi_3^{(23)}}{{\tau_{23}}^3}\frac{\left\langle\vec{\xi}^{(23)},\dot{\vec{u}}^{(3)}\right\rangle}{c^3}-\frac{\dot{u}_3^{(3)}}{c\tau_{23}}\right)-\frac{e_2^2}{m_2c^3}\ddot{u}_3^{(2)}\right]\sin\lambda_2+$$
$$+\left[\frac{e_2e_1}{m_2c^4}\frac{x_3^{(2)}}{{\tau_{21}}^3}+\frac{e_2e_3}{m_2c^3}\left(\frac{\xi_3^{(23)}}{{\tau_{23}}^3}\frac{\left\langle\vec{\xi}^{(23)},\dot{\vec{u}}^{(3)}\right\rangle}{c^3}-\frac{\dot{u}_3^{(3)}}{c\tau_{23}}\right)-\frac{e_2^2}{m_2c^3}\ddot{u}_3^{(2)}\right]\eta_2\sin\lambda_2\le$$
$$\le\frac{e_2e_1}{m_2c^4}\frac{u_1^{(2)}}{{\tau_{21}}^3}-3\frac{e_2e_1}{m_2c^4}\frac{x_1^{(2)}\dot{\tau}_{21}}{{\tau_{21}}^4}+$$
$$+\frac{e_2e_3}{m_2c^3}\left(\frac{\dot{\xi}_1^{(23)}}{{\tau_{23}}^3}\frac{\left\langle\vec{\xi}^{(23)},\dot{\vec{u}}^{(3)}\right\rangle}{c^3}-\frac{3\xi_1^{(23)}\dot{\tau}_{23}}{{\tau_{23}}^4}\frac{\left\langle\vec{\xi}^{(23)},\dot{\vec{u}}^{(3)}\right\rangle}{c^3}+\frac{\xi_1^{(23)}}{{\tau_{23}}^3}\frac{\left\langle\dot{\vec{\xi}}^{(23)},\dot{\vec{u}}^{(3)}\right\rangle+\left\langle\vec{\xi}^{(23)},\ddot{\vec{u}}^{(3)}\right\rangle}{c^3}-\frac{\ddot{u}_1^{(3)}}{c\tau_{23}}+\frac{\dot{u}_1^{(3)}\dot{\tau}_{23}}{c{\tau_{23}}^2}\right)-\frac{e_2^2}{m_2c^3}\dddot{u}_1^{(2)}+$$
$$+\frac{|e_2e_1|}{m_2c}\left[\frac{1}{{\widehat{\rho}_2}^{\,2}}+\frac{\bar{c}\omega_3\left(\breve{\rho}_2+\breve{\rho}_3+3\Delta\right)}{c^3\Delta^2}+\frac{{\omega_2}^2R_2}{c^2}\right](\Phi_2+Y_2)e^{\mu(t-T_k)}+$$
$$+\frac{e_2e_1}{m_2c^4}\frac{u_2^{(2)}}{{\tau_{21}}^3}-3\frac{e_2e_1}{m_2c^4}\frac{x_2^{(2)}\dot{\tau}_{21}}{{\tau_{21}}^4}+$$
$$+\frac{e_2e_3}{m_2c^3}\left(\frac{\dot{\xi}_2^{(23)}}{{\tau_{23}}^3}\frac{\left\langle\vec{\xi}^{(23)},\dot{\vec{u}}^{(3)}\right\rangle}{c^3}-\frac{3\xi_2^{(23)}\dot{\tau}_{23}}{{\tau_{23}}^4}\frac{\left\langle\vec{\xi}^{(23)},\dot{\vec{u}}^{(3)}\right\rangle}{c^3}+\frac{\xi_2^{(23)}}{{\tau_{23}}^3}\frac{\left\langle\dot{\vec{\xi}}^{(23)},\dot{\vec{u}}^{(3)}\right\rangle+\left\langle\vec{\xi}^{(23)},\ddot{\vec{u}}^{(3)}\right\rangle}{c^3}-\frac{\ddot{u}_2^{(3)}}{c\tau_{23}}+\frac{\dot{u}_2^{(3)}\dot{\tau}_{23}}{c{\tau_{23}}^2}\right)-\frac{e_2^2}{m_2c^3}\dddot{u}_2^{(2)}+$$
$$+\left[\frac{e_2e_1}{m_2c^4}\frac{x_2^{(2)}}{{\tau_{21}}^3}+\frac{e_2e_3}{m_2c^3}\left(\frac{\xi_2^{(23)}}{{\tau_{23}}^3}\frac{\left\langle\vec{\xi}^{(23)},\dot{\vec{u}}^{(3)}\right\rangle}{c^3}-\frac{\dot{u}_2^{(3)}}{c\tau_{23}}\right)-\frac{e_2^2}{m_2c^3}\ddot{u}_2^{(2)}\right](\Phi_2+Y_2)e^{\mu(t-T_k)}+$$
$$+\frac{e_2e_1}{m_2c^4}\frac{u_3^{(2)}}{{\tau_{21}}^3}-3\frac{e_2e_1}{m_2c^4}\frac{x_3^{(2)}\dot{\tau}_{21}}{{\tau_{21}}^4}+$$
$$+\frac{e_2e_3}{m_2c^3}\left(\frac{\dot{\xi}_3^{(23)}}{{\tau_{23}}^3}\frac{\left\langle\vec{\xi}^{(23)},\dot{\vec{u}}^{(3)}\right\rangle}{c^3}-\frac{3\xi_3^{(23)}\dot{\tau}_{23}}{{\tau_{23}}^4}\frac{\left\langle\vec{\xi}^{(23)},\dot{\vec{u}}^{(3)}\right\rangle}{c^3}+\frac{\xi_3^{(23)}}{{\tau_{23}}^3}\frac{\left\langle\dot{\vec{\xi}}^{(23)},\dot{\vec{u}}^{(3)}\right\rangle+\left\langle\vec{\xi}^{(23)},\ddot{\vec{u}}^{(3)}\right\rangle}{c^3}-\frac{\ddot{u}_3^{(3)}}{c\tau_{23}}+\frac{\dot{u}_3^{(3)}\dot{\tau}_{23}}{c{\tau_{23}}^2}\right)-\frac{e_2^2}{m_2c^3}\dddot{u}_3^{(2)}+$$
$$+\left[\frac{e_2e_1}{m_2c^4}\frac{x_3^{(2)}}{{\tau_{21}}^3}+\frac{e_2e_3}{m_2c^3}\left(\frac{\xi_3^{(23)}}{{\tau_{23}}^3}\frac{\left\langle\vec{\xi}^{(23)},\dot{\vec{u}}^{(3)}\right\rangle}{c^3}-\frac{\dot{u}_3^{(3)}}{c\tau_{23}}\right)-\frac{e_2^2}{m_2c^3}\ddot{u}_3^{(2)}\right]Y_2e^{\mu(t-T_k)};$$

$$\left|\frac{d^2B_{r_2}^{(k)}}{dt^2}\right|=\left|\frac{dG_{r_2}}{dt}\right|\le\frac{|e_2e_1|}{m_2}\left[\frac{\widetilde{c}}{c^4\Delta^3}+\frac{3\rho_2}{c^4\Delta^4}+\frac{8\omega_3\bar{c}}{c^4\Delta^2}+\frac{2{\omega_3}^2\bar{c}}{c^4\Delta}+\frac{{\omega_2}^3R_2}{c^3}\right]e^{\mu(t-T_k)}+\frac{|e_2e_1|}{m_2c}\left[\frac{1}{{\widehat{\rho}_2}^{\,2}}+\frac{\bar{c}\omega_3\left(\breve{\rho}_2+\breve{\rho}_3+3\Delta\right)}{c^3\Delta^2}+\frac{{\omega_2}^2R_2}{c^2}\right](\Phi_2+Y_2)e^{\mu(t-T_k)}+$$
$$+\frac{|e_2e_1|}{m_2}\left[\frac{\bar{c}}{c^4\Delta^3}+\frac{3\rho_2}{c^4\Delta^4}+\frac{8\omega_3\bar{c}}{c^4\Delta^2}+\frac{2{\omega_3}^2\bar{c}}{c^4\Delta}+\frac{{\omega_2}^3R_2}{c^3}\right]e^{\mu(t-T_k)}+\frac{|e_2e_1|}{m_2c}\left[\frac{1}{{\widehat{\rho}_2}^{\,2}}+\frac{\bar{c}\omega_3\left(\breve{\rho}_2+\breve{\rho}_3+3\Delta\right)}{c^3\Delta^2}+\frac{{\omega_2}^2R_2}{c^2}\right](\Phi_2+Y_2)e^{\mu(t-T_k)}+$$
$$+\frac{|e_2e_1|}{m_2}\left[\frac{\widetilde{c}}{c^4\Delta^3}+\frac{3\rho_2}{c^4\Delta^4}+\frac{8\omega_3\bar{c}}{c^4\Delta^2}+\frac{2{\omega_3}^2\bar{c}}{c^4\Delta}\frac{{\omega_2}^3R_2}{c^3}\right]e^{\mu(t-T_k)}+\frac{|e_2e_1|}{m_2c}\left[\frac{1}{{\widehat{\rho}_2}^{\,2}}+\frac{\bar{c}\omega_3\left(\breve{\rho}_2+\breve{\rho}_3+3\Delta\right)}{c^3\Delta^2}+\frac{{\omega_2}^2R_2}{c^2}\right]Y_2e^{\mu(t-T_k)}\le{\omega_2}^2R_2e^{\mu(t-T_k)}.$$

Appendix E. Upper Bounds for Partial Derivatives

Since $\varphi_{23} = \varphi_2(t) - \varphi_3(t - \tau_{23})$ and $\left|\left\langle \vec{\xi}^{(23)}, \dot{\vec{u}}^{(3)} \right\rangle\right| \leq \bar{c}c\tau_{23}\omega_3 e^{\mu T}$, one obtains

$$\begin{aligned}
&\left|\frac{\partial G_{r_2}}{\partial \rho_2}\right| \leq \frac{|e_2e_1|}{m_2}\left|\frac{2}{c\rho_2{}^3} + \frac{1}{c^3}\frac{\bar{c}c\tau_{23}\omega_3 e^{\mu(t-T_k)}}{c^3\tau_{23}{}^3}\right| \leq \frac{|e_2e_1|}{m_2}\left(\frac{2}{c\widehat{\rho}_2{}^3} + \frac{\bar{c}\omega_3}{c^5\Delta^2}\right)e^{\mu T} \equiv \frac{\partial \overline{G}_{r_2}}{\partial \rho_2};\\
&\left|\frac{\partial G_{r_2}(r_2,\phi_2,\eta_2,r_3,\phi_3,\eta_3)}{\partial r_2}\right| = 0\,;\left|\frac{\partial G_{r_2}}{\partial \rho_3}\right| \leq \frac{|e_2e_3|\omega_3}{m_2}\frac{\bar{c}+c(\Phi_3+Y_3)\Delta}{c^6\Delta^2}e^{\mu T} \equiv \frac{\partial \overline{G}_{r_2}}{\partial \rho_3};\\
&\left|\frac{\partial G_{r_2}}{\partial \dot{r}_3}\right| \leq \frac{|e_2e_3|}{m_2}\frac{2(\breve{\rho}_2+\breve{\rho}_3)^2+\Delta^2}{c^6\Delta^3} \equiv \frac{\partial \overline{G}_{r_2}}{\partial r_3};\\
&\left|\frac{\partial G_{r_2}}{\partial \varphi_2}\right| \leq \frac{|e_2e_3|}{m_2}\omega_3\bar{c}\left(\frac{\breve{\rho}_3 e^{\mu T}}{c^5\Delta^2} + \frac{\breve{\rho}_2+\breve{\rho}_3}{c^3} + \frac{\breve{\rho}_2}{c^3\Delta^3} + \frac{1}{c^4\Delta}\right) \equiv \frac{\partial \overline{G}_{r_2}}{\partial \varphi_2};\\
&\left|\frac{\partial G_{r_2}}{\partial \varphi_3}\right| \leq \frac{|e_2e_1|}{m_2}\left[\frac{\breve{\rho}_2+\breve{\rho}_3}{c^3}\frac{\bar{c}\omega_3 e^{\mu T}}{c^2\Delta^2} + \frac{\breve{\rho}_2+\breve{\rho}_3}{c^3} + \frac{\breve{\rho}_2\omega_3\bar{c}}{c^3\Delta^3} + \frac{\omega_3\bar{c}}{c^4\Delta}\right] \equiv \frac{\partial \overline{G}_{r_2}}{\partial \varphi_3};\\
&\left|\frac{\partial G_{r_2}(r_2,\phi_2,\eta_2,r_3,\phi_3,\eta_3)}{\partial \phi_2}\right| = 0;\ \left|\frac{\partial G_{r_2}(r_2,\phi_2,\eta_2,r_3,\phi_3,\eta_3)}{\partial \phi_3}\right| = 0;\\
&\left|\frac{\partial G_{r_2}}{\partial \lambda_2}\right| \leq \frac{|e_2e_3|}{m_2}e^{\mu T}\bar{c}\omega_3\left(\frac{\breve{\rho}_2+\underset{\leftarrow 3}{\breve{\rho}}}{c^3}\frac{\Delta+\breve{\rho}_2}{c^3\Delta^3} + \frac{1}{c^4\Delta}\right) = \frac{\partial \overline{G}_{r_2}}{\partial \lambda_2};\ \left|\frac{\partial G_{r_2}(r_2,\phi_2,\eta_2,r_3,\phi_3,\eta_3)}{\partial \eta_2}\right| = 0;\\
&\left|\frac{\partial G_{r_2}}{\partial \lambda_3}\right| \leq \frac{|e_2e_1|}{m_2}\frac{\bar{c}\omega_3}{c^4\Delta}\left(\frac{(\breve{\rho}_2+\breve{\rho}_3)(\Delta e^{\mu T}+2\breve{\rho}_2)}{c^2\Delta^2} + 1\right) \equiv \frac{\partial \overline{G}_{r_2}}{\partial \lambda_3};\\
&\left|\frac{\partial G_{r_2}(r_2,\phi_2,\eta_2,r_3,\phi_3,\eta_3)}{\partial \eta_3}\right| = 0;\left|\frac{\partial G_{r_2}}{\partial \ddot{r}_2}\right| \leq \frac{e_2^2}{m_2c^3} \equiv \frac{\partial \overline{G}_{r_2}}{\partial \ddot{r}_2}.
\end{aligned}$$

The upper bounds for $\frac{\partial G_{r_3}}{\partial \rho_2}, \ldots, \frac{\partial G_{r_3}}{\partial \ddot{r}_3}, \ldots, \frac{\partial G_{\eta_3}}{\partial \ddot{r}_3}$ can be obtained in a similar way.

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