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On Some Properties of a Complete Quadrangle

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Abstract: In this paper, we study the properties of a complete quadrangle in the Euclidean plane. The proofs are based on using rectangular coordinates symmetrically on four vertices and four parameters a, b, c, d. Here, many properties of the complete quadrangle known from earlier research are proved using the same method, and some new results are given.

Keywords: complete quadrangle; diagonal triangle; anticenter

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1. Introduction

If four points are joined in pairs by six distinct lines, they are called the *vertices* of a *complete quadrangle*, and the lines are its six *sides*. Two sides are said to be *opposite* if they have no common vertex. The study of the geometry of the complete quadrangle has a long history and there are numerous articles in which the properties of quadrangles have been studied. In this paper, we deal with the properties of quadrangles related to the center and anticenter of the quadrangle, the diagonal triangle of the quadrangle, and isogonality with respect to the four triangles formed by the vertices of the quadrangle. These properties were studied in the literature using a number of various methods [1–12].

Our approach in this paper uses a novel method which is applicable to studying and extending the known properties of a quadrangle. We put the complete quadrangle into such a coordinate system that its circumscribed hyperbola is rectangular. We use this method to prove the 12 theorems already published in aforementioned papers and to derive two new original theorems, Theorems 8 and 14, which to our knowledge were not yet published in the literature. Thus, our method allows one to study the properties of quadrangles in a more unified way.

In our former work in [13], we analyzed a complete quadrilateral in a similar way. A complete quadrilateral is a set of four lines (sides of the quadrilateral), where none of two lines are parallel and none of the three are concurrent. Using the fact that a unique parabola can be inscribed on each quadrilateral, the coordinate system was chosen so that the parabola has the equation $y^2 = 4x$. The sides of the quadrilateral are given by $ay = x + a^2$, $by = x + b^2$, $cy = x + c^2$, $dy = x + d^2$ where a, b, c, d are real numbers. The coordinate system chosen in this way is suitable for studying quadrilaterals, but not for studying quadrangles.

As in [14], we proved:

Lemma 1. *For each quadrangle for which the opposite sides are not perpendicular, the rectangular hyperbola can be circumscribed.*

Therefore, we choose the coordinate system for studying complete quadrangles such that circumscribed hyperbola of the complete quadrangle is given by $xy = 1$. In the same paper, we studied the quadruples of orthopoles.

2. Methods

Let $ABCD$ be a complete quadrangle and $\mathcal{H}$ be a rectangular hyperbola circumscribed to it. With the suitable choice of the coordinate system, it can be achieved that $\mathcal{H}$ has the equation $xy = 1$ and the vertices of the quadrangle are of the form

$$A = \left(a, \frac{1}{a}\right), B = \left(b, \frac{1}{b}\right), C = \left(c, \frac{1}{c}\right), D = \left(d, \frac{1}{d}\right), \tag{1}$$

where $a, b, c, d \neq 0$.

Let s, q, r, p be elementary symmetric functions in four variables a, b, c, d:

$$s = a + b + c + d, \quad q = ab + ac + ad + bc + bd + cd,$$
$$r = abc + abd + acd + bcd, \quad p = abcd.$$

The centroid of the quadrangle $ABCD$ is of the form

$$G = \left(\frac{s}{4}, \frac{r}{4p}\right). \tag{2}$$

The sides of $ABCD$ have the equations:

$$\begin{aligned} &AB \ldots x + aby = a + b, \quad AC \ldots x + acy = a + c, \quad AD \ldots x + ady = a + d \\ &BC \ldots x + bcy = b + c, \quad BD \ldots x + bdy = b + d, \quad CD \ldots x + cdy = c + d. \end{aligned} \tag{3}$$

The choice of the equation of the hyperbola $\mathcal{H}$, i.e., the coordinates of the vertices, enables us to prove the claims in a simple way using an analytical method. The calculations are elementary and mostly very short.

The paper is organized in such a way that we first prove a property, and then state it in a theorem. After the theorem, we point out whether the claim is previously known from the literature or is our original contribution.

3. Results

3.1. The Center and Anticenter of the Quadrangle ABCD

In this section, we study the Euler circles of four triangles of the quadrangle $ABCD$, and define its center and anticenter. The circle with the equation

$$2abc\left(x^2 + y^2\right) + [1 - abc(a + b + c)]x - \left(a^2b^2c^2 - ab - ac - bc\right)y = 0$$

passes through the midpoint $(\frac{1}{2}(a + b), \frac{1}{2ab}(a + b))$ of points A and B. Similarly, it passes through the midpoints of A, C, i.e., B, C, so it is Euler's circle $\mathcal{N}_d$ of the triangle ABC. It obviously passes through the origin O. Analogously, the same is valid for Euler's circles $\mathcal{N}_c$, $\mathcal{N}_b$, and $\mathcal{N}_a$ of the triangles ABD, ACD, and BCD. Hence, we have just proved the following statement:

Theorem 1. *Euler's circles of the triangles BCD, ACD, ABD, and ABC of the complete quadrangle with the circumscribed rectangular hyperbola passes through the center of the hyperbola.*

The theorem is coming from [3].

There are several names for the point O in the literature. In this paper, we call it the *center* of the quadrangle $ABCD$. The point $O' = (\frac{s}{2}, \frac{r}{2p})$, symmetric to the point O with respect to the centroid G, we call the *anticenter* of the quadrangle $ABCD$. The asymptotes $\mathcal{X}$ and $\mathcal{Y}$ of the hyperbola $\mathcal{H}$ are the *axes* of the quadrangle $ABCD$.

The center N_d of the circle $\mathcal{N}_d$, i.e., Euler's center of the triangle ABC, is the point

$$N_d = \left(\frac{1}{4}\left(a+b+c-\frac{1}{abc}\right), \frac{1}{4}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-abc\right)\right). \tag{4}$$

The Euler's centers N_a, N_b, N_c of the triangles BCD, ACD, ABD are of similar forms. The distance from N_d to the origin O fulfills

$$\begin{aligned} ON_d^{\,2} &= \left(\frac{1}{4abc}\right)^2 [abc(a+b+c)-1]^2 + (ab+ac+bc-a^2b^2c^2)^2 \\ &= \left(\frac{1}{4abc}\right)^2 (a^2b^2+1)(a^2c^2+1)(b^2c^2+1). \end{aligned}$$

Hence, Euler's circle of the triangle ABC has the radius $\frac{1}{4}|\frac{d}{p}|\sqrt{(a^2b^2+1)(a^2c^2+1)(b^2c^2+1)}$. The other three radii of the Euler's circles of the other three triangles look quite similar, as can be seen in [11]. Because of that, the radius ρ_d of the circumscribed circle of the triangle ABC is given within following analogous formulae

$$\rho_a = \frac{1}{2}\left|\frac{a}{p}\right|\sqrt{\lambda'\mu'\nu'}, \quad \rho_b = \frac{1}{2}\left|\frac{b}{p}\right|\sqrt{\lambda'\mu\nu}, \quad \rho_c = \frac{1}{2}\left|\frac{c}{p}\right|\sqrt{\lambda\mu'\nu}, \quad \rho_d = \frac{1}{2}\left|\frac{d}{p}\right|\sqrt{\lambda\mu\nu'},$$

where ρ_a, ρ_b, ρ_c are the radii of the circumscribed circles of the triangles BCD, ACD, ABD using the following notations

$$\begin{aligned} \lambda &= a^2b^2+1, \quad \mu = a^2c^2+1, \quad \nu = a^2d^2+1, \\ \lambda' &= c^2d^2+1, \quad \mu' = b^2d^2+1, \quad \nu' = b^2c^2+1, \end{aligned} \tag{5}$$

where $\lambda, \mu, \nu, \lambda', \mu', \nu' > 0$. The parameters (5) appear in formulae for the lengths of the sides of the quadrangle $ABCD$. Indeed, for the points A and B, we obtain

$$AB^2 = (a-b)^2 + \left(\frac{1}{a}-\frac{1}{b}\right)^2 = \left(\frac{a-b}{ab}\right)^2 (a^2b^2+1) = \left(\frac{a-b}{ab}\right)^2 \lambda,$$

i.e., $AB = \left|\frac{a-b}{ab}\right|\sqrt{\lambda}$. The other five analogous statements are also valid

$$AC = \left|\frac{a-c}{ac}\right|\sqrt{\mu}, AD = \left|\frac{a-d}{ad}\right|\sqrt{\nu}, BC = \left|\frac{b-c}{bc}\right|\sqrt{\nu'}, BD = \left|\frac{b-d}{bd}\right|\sqrt{\mu'}, CD = \left|\frac{c-d}{cd}\right|\sqrt{\lambda'}.$$

From these equalities, the next equalities follow

$$AB \cdot CD = \left|\frac{(a-b)(c-d)}{p}\right|\sqrt{\lambda\lambda'}, \quad AC \cdot BD = \left|\frac{(a-c)(b-d)}{p}\right|\sqrt{\mu\mu'},$$
$$AD \cdot BC = \left|\frac{(a-d)(b-c)}{p}\right|\sqrt{\nu\nu'}.$$

For the coordinates of the point N_d from (4), it proves that

$$\left(x-\frac{s}{4}\right)\left(y-\frac{r}{4p}\right) = \frac{1}{16p}(p+1)^2.$$

The same is also valid for N_a, N_b, N_c. Therefore, we have proved the result:

Theorem 2. *The centroid G of the quadrangle $ABCD$ is the center of the quadrangle $N_aN_bN_cN_d$, where N_a, N_b, N_c, N_d are the centers of Euler circles BCD, ACD, ABD, ABC, respectively, and the quadrangles $ABCD$ and $N_aN_bN_cN_d$ have the parallel axes.*

This result is coming from [7,11]. Because the midpoints AD, BD, CD are symmetric to the midpoints BC, AC, AB with respect to the centroid G, the circle incident to the midpoints of AD, BD, CD is symmetric to the Euler circle $\mathcal{N}_d$ of the triangle ABC with respect to the centroid G. Hence, that circle is incident to anticenter O' because the circle $\mathcal{N}_d$ is incident to O. We have proved the following:

Theorem 3. *Circles incident to the midpoints of three sides AD, BD, CD; AC, BC, CD; AB, BC, BD; AB, AC, AD are passing through the anticenter O'.*

The result is also given in [1,10].

The line AB has the slope $-\frac{1}{ab}$, and the connecting line of the origin and the midpoint of AB has the slope $\frac{1}{ab}$, so these lines are antiparallel with respect to the coordinate axes. The same is valid for any side of the quadrangle $ABCD$. We showed the result:

Theorem 4. *The angle of any two sides of the quadrangle is opposite to the angle of the connecting lines of the midpoints of these sides and the center of $ABCD$.*

This result was also given in [3,12]. Let us study the points

$$H_a = \left(-\frac{1}{bcd}, -bcd\right), H_b = \left(-\frac{1}{acd}, -acd\right), H_c = \left(-\frac{1}{abd}, -abd\right), H_d = \left(-\frac{1}{abc}, -abc\right). \tag{6}$$

The line with the equation $abx - y = abc - \frac{1}{c}$ is perpendicular to the line AB from (3) and it is incident to C and H_d, so the line CH_d is height from C of the triangle ABC. Similarly, the lines AH_d and BH_d are the heights from the vertices A and B of the triangle ABC. Therefore, H_d is the orthocenter of that triangle. Hence, we showed that the following is valid:

Theorem 5. *The orthocenters H_a, H_b, H_c, H_d of the triangles BCD, ACD, ABD, ABC, respectively, are incident to the rectangular hyperbola $\mathcal{H}$.*

This statement has been proven in [3], and it also proves the converse of Lemma 2 from [14].

As the orthocenters H_a, H_b, H_c, H_d are incident to hyperbola $\mathcal{H}$, its center O is the center of the quadrangle $H_aH_bH_cH_d$. Thus, we have proved:

Theorem 6. *Quadrangles $ABCD$ and $H_aH_bH_cH_d$ have the same center.*

This result also appears in [11].

If the point D coincides with H_d, then $d = \frac{1}{abc}$, $p = -1$, and the quadrangle $ABCD$ is the orthocentric quadrangle (see [14]).

3.2. A Diagonal Triangle of the Quadrangle ABCD

Diagonal points $U = AB \cap CD$, $V = AC \cap BD$, $W = AD \cap BC$ of the quadrangle $ABCD$ are given by

$$U = \left(\frac{ab(c+d) - cd(a+b)}{ab - cd}, \frac{a+b-c-d}{ab-cd}\right), \quad V = \left(\frac{ac(b+d) - bd(a+c)}{ac - bd}, \frac{a+c-b-d}{ac-bd}\right),$$
$$W = \left(\frac{ad(b+c) - bc(a+d)}{ad - bc}, \frac{a+d-b-c}{ad-bc}\right).$$

These points can be written in the shorter form

$$U = \left(\frac{u'}{u}, \frac{u''}{u}\right), \quad V = \left(\frac{v'}{v}, \frac{v''}{v}\right), \quad W = \left(\frac{w'}{w}, \frac{w''}{w}\right),$$

where

$$\begin{aligned} u &= ab - cd, \quad u' = ab(c+d) - cd(a+b), \quad u'' = a+b-c-d, \\ v &= ac - bd, \quad v' = ac(b+d) - bd(a+c), \quad v'' = a+c-b-d, \\ w &= ad - bc, \quad w' = ad(b+c) - bc(a+d), \quad w'' = a+d-b-c. \end{aligned}$$

The following equalities are valid

$$u'v'' + u''v' = 2uv, \quad u'w'' + u''w' = 2uw, \quad v'w'' + v''w' = 2vw.$$

Therefore, the lines $\mathcal{U}$, $\mathcal{V}$, $\mathcal{W}$ with equations

$$u''x + u'y = 2u, \quad v''x + v'y = 2v, \quad w''x + w'y = 2w$$

are incident to the pairs of points V, W; U, W; U, V, respectively. So, they are the diagonals of the quadrangle $ABCD$. Hence, their equations are

$$\begin{aligned} \mathcal{U} \quad \ldots \quad & (a+b-c-d)x + [ab(c+d) - cd(a+b)]y = 2(ab-cd), \\ \mathcal{V} \quad \ldots \quad & (a+c-b-d)x + [ac(b+d) - bd(a+c)]y = 2(ac-bd), \\ \mathcal{W} \quad \ldots \quad & (a+d-b-c)x + [ad(b+c) - bc(a+d)]y = 2(ad-bc). \end{aligned}$$

The centroid G_{UVW} of the triangle UVW is the point

$$G_{UVW} = \left(\frac{u'vw + uv'w + uvw'}{3uvw}, \frac{u''vw + uv''w + uvw''}{3uvw} \right).$$

The heights from vertices U and V of the diagonal triangle UVW have the equations

$$uu'x - uu''y = u'^2 - u''^2, \quad vv'x - vv''y = v'^2 - v''^2.$$

For their intersection point (x, y), the equalities

$$\begin{aligned} uv(u'v'' - u''v')x &= u'^2vv'' - uu''v'^2 + u''v''(uv'' - u''v), \\ uv(u'v'' - u''v')y &= u'v'(u'v - uv') + uu'v''^2 - u''^2vv' \end{aligned}$$

are valid. However, it can be checked that

$$\begin{aligned} u'v'' - u''v' &= 2(a-d)(b-c)w, \\ u'^2vv'' - uu''v'^2 &= (a-d)(b-c)(u'vw + uv'w + uvw'), \qquad (7) \\ uv'' - u''v &= (a-d)(b-c)w'', \qquad (8) \\ u'v - uv' &= (a-d)(b-c)w', \qquad (9) \\ uu'v''^2 - u''^2vv' &= (a-d)(b-c)(u''vw + uv''w + uvw'') \end{aligned}$$

are valid. Hence, the orthocenter of the triangle UVW is the point

$$H_{UVW} = \left(\frac{u'vw + uv'w + uvw' + u''v''w''}{2uvw}, \frac{u''vw + uv''w + uvw'' + u'v'w'}{2uw} \right).$$

The centroid, orthocenter, and circumcenter O_{UVW} of the triangle UVW fulfill the equality $2O_{UVW} + H_{UVW} = 3G_{UVW}$, out of which we obtain

$$O_{UVW} = \left(\frac{u'vw + uv'w + uvw' - u''v''w''}{4uvw}, \frac{u''vw + uv''w + uvw'' - u'v'w'}{4uvw} \right).$$

Now let us study the circle $\mathcal{K}_{UVW}$ with the center O_{UVW} and the equation

$$2uvw(x^2+y^2)-(u'vw+uv'w+uvw'-u''v''w'')x-(u''vw+uv''w+uvw''-u'v'w')y=0. \tag{10}$$

We will show the circumscribed circle of the triangle UVW. We will also show that U is incident to this circle, and it is proved by the equality

$$2vw(u'^2+u''^2)-(u'vw+uv'w+uvw'-u''v''w'')u'-(u''vw+uv''w+uvw''-u'v'w')u''=0$$

that can be written in the form

$$u'w(u'v-uv')+u'w'(u''v'-uv)-u''w(uv''-u''v)+u''w''(u'v''-uv)=0$$

and it is valid because of (7) and (8) and the equalities

$$\begin{aligned} u''v'-uv &= -(a-d)(b-c)w, & (11)\\ u'v''-uv &= (a-d)(b-c)w. & (12)\end{aligned}$$

Theorem 7. *The circumscribed circle of the diagonal triangle UVW of the quadrangle $ABCD$ is incident to its center O.*

The same result can be found in [2,3,6,8,10].

The line $\mathcal{U}$ has the equation $u''x+u'y=2u$ and the normal from O to this line is given by $u'x-u''y=0$. The intersection point of these two lines is the point

$$\left(\frac{2uu''}{u'^2+u''^2},\frac{2uu'}{u'^2+u''^2}\right). \tag{13}$$

Out of the equalities (8) and (12), and (9) and (11), the next equalities follow

$$(uv''-u''v)w=(u'v''-uv)w'',\quad (uv'-u'v)w=(u''v'-uv)w'$$

that can be written in the form

$$uv''w+uvw''-u'v''w''=u''vw,\quad uv'w+uvw'-u''v'w'=u'vw. \tag{14}$$

The expression

$$(u''vw+uv''w+uvw''-u'v'w')u''+(u'vw+uv'w+uvw'-u''v''w'')u'$$

can be written as

$$vw(u'^2+u''^2)+u''(uv''w+uvw''-u'v''w'')+u'(uv'w+uvw'-u''v'w'),$$

and because of (14), which is equal to $vw(u'^2+u''^2)+vwu''^2+vwu'^2=2vw(u'^2+u''^2)$. This means that line with the equation

$$\mathcal{W}_o\ldots(u''vw+uv''w+uvw''-u'v'w')x+(u'vw+uv'w+uvw'-u''v''w'')y=4uvw \tag{15}$$

is incident to the point (13), the pedal of the normal to the line $\mathcal{U}$ from the point O. Because of the symmetry, it is incident to the pedals of the normal to the line $\mathcal{V}$ and $\mathcal{W}$ from the point O, respectively. Hence, the line $\mathcal{W}_o$ in (15) is the Wallace's line of the point O with respect to the triangle UVW. Therefore, we proved our original statement, as can be seen in Figure 1:

Theorem 8. *The Wallace's line of the center O with respect to the diagonal triangle UVW and the connecting line of the points O_{UVW} and O form equal angles with the asymptotes $\mathcal{X}$ and $\mathcal{Y}$ of the hyperbola $\mathcal{H}$.*

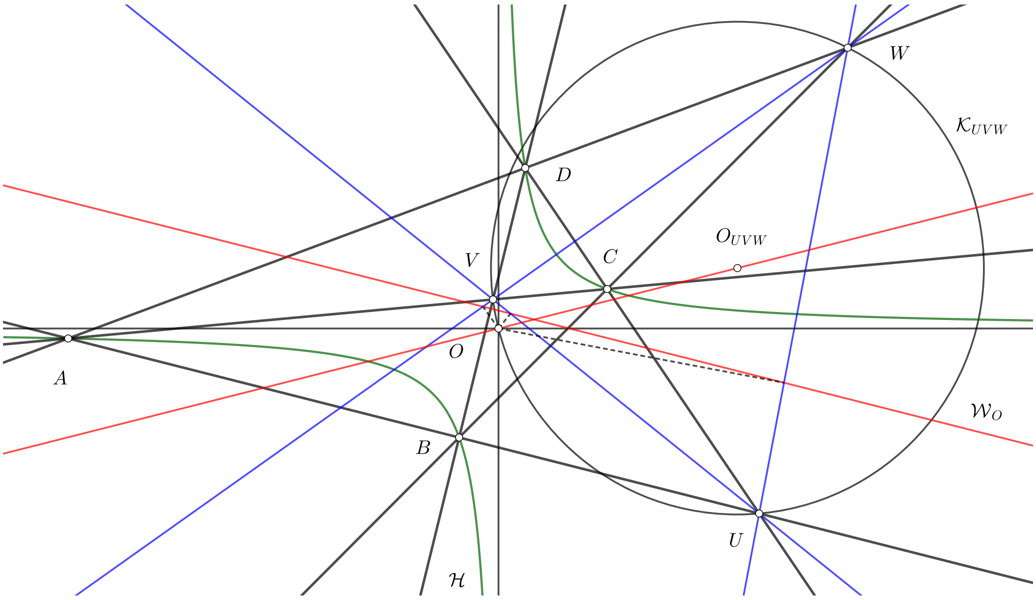

Figure 1. The Wallace's line $\mathcal{W}_O$ of the center O with respect to the triangle UVW and the line OO_{UVW} form equal angles with the asymptotes of $\mathcal{H}$.

Namely, their slopes are opposite.

The line through the midpoint $(\frac{a+b}{2}, \frac{a+b}{2ab})$ of the side AB and parallel to the line CD has the equation $x + cdy - \frac{a+b}{2ab}(ab + cd) = 0$ and it is incident to the point

$$U_o = \left((ab + cd)\frac{u''}{2u}, (ab + cd)\frac{u'}{2pu}\right)$$

because

$$abu'' + u' - (a + b)u = 0. \tag{16}$$

Because of the symmetry of the coordinates of this point on pairs a, b and c, d, it follows that the line incident to the midpoint of CD and parallel to the side AB is also incident to U_o. The midpoint of AB and the point U_o are lying on the circle given by

$$2pu(x^2 + y^2) + [p(u' - su) + u']x + [p^2u'' + c^2d^2(c + d) - a^2b^2(a + b)]y = 0.$$

This circle is incident to the midpoint of CD and obviously to the point O. There are two more such circles obtained in an analogous way. As it is stated in [1,3], the following is valid:

Theorem 9. *The circles incident to the midpoints of AB, CD, and the point U_o; AC, BD, and V_0; AD, BC, and W_0 are incident to O.*

The triangles BCD and ACD have centroids $G_a = \left(\frac{1}{3}(b + c + d), \frac{1}{3}(\frac{1}{b} + \frac{1}{c} + \frac{1}{d})\right)$, $G_b = \left(\frac{1}{3}(a + c + d), \frac{1}{3}(\frac{1}{a} + \frac{1}{c} + \frac{1}{d})\right)$ and their connecting line G_aG_b has the equation $3cdx + 3py = cds + ab(c + d)$. Analogously, the line G_cG_d has the equation $3abx + 3py = abs + cd(a + b)$. The intersection point $U_g = G_aG_b \cap G_cG_d$ is of the form

$$U_g = \left(\frac{s}{3} - \frac{u'}{3u}, \frac{c + d}{3cd} + \frac{u'}{3abu}\right).$$

The orthocenters H_a and H_b from (6) have a connecting line H_aH_b with the equation $cdpx + y = -cd(a + b)$, and analogously, the line H_cH_d has the equation $abpx + y = -ab(c + d)$. The intersection point $U_h = H_aH_b \cap H_cH_d$ is

$$U_h = \left(-\frac{u'}{pu}, -\frac{pu''}{u}\right).$$

Let us remember from [14] that the circumcenter of the triangle ABC is the point

$$O_d = \left(\frac{1}{2}\left(a + b + c + \frac{1}{abc}\right), \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + abc\right)\right). \tag{17}$$

The circumcenters O_a and O_b with forms analogous to (17) have the connecting line O_aO_b with the equation $cdx - y = \frac{1}{2cd}(c + d)(c^2d^2 + 1)$, and analogously, the line O_cO_d has the equation $abx - y = \frac{1}{2ab}(a + b)(a^2b^2 + 1)$. For the intersection point $U_o = O_aO_b \cap O_cO_d$, we obtain the form

$$U_o = \left(\frac{1}{2pu}(psu - pu' + u'), \frac{1}{2pu}[ab(c + d)u + cdu' + p^2u'']\right).$$

Out of the terms for U_g, U_h, and U_o, it is easy to check that the equality $U_h + 2U_o = 3U_g$ is valid, i.e., $U_h - U_g = 2(U_g - U_o)$ or $U_gU_h = 2U_oU_g$, i.e., $U_oU_g : U_gU_h = 1 : 2$. The same is valid for the analogous intersections. So, we have proved the result that can be found in [9], where Myakishev addressed it to J. Ganin:

Theorem 10. *If G_a, G_b, G_c, G_d are centroids, H_a, H_b, H_c, H_d are orthocenters and O_a, O_b, O_c, O_d are the circumcenters of the triangles BCD, ACD, ABD, ABC in the quadrangle $ABCD$, and if U_g, V_g, W_g; U_h, V_h, W_h and U_o, V_o, W_o represent the diagonal points of the quadrangles $G_aG_bG_cG_d$, $H_aH_bH_cH_d$, and $O_aO_bO_cO_d$, respectively, then the triples of points U_g, U_h, U_o; V_g, V_h, V_o; W_g, W_h, W_o are collinear and $U_oU_g : U_gU_h = V_oV_g : V_gV_h = W_oW_g : W_gW_h = 1 : 2$ is valid.*

3.3. Isogonality with Respect to the Triangles BCD, ACD, ABD, ABC

If two lines $\mathcal{L}$ and $\mathcal{L}'$ have slopes $\frac{m}{n}$ and $\frac{m'}{n'}$, then for the oriented angle $\angle(\mathcal{L}, \mathcal{L}')$, the following formula is valid

$$tg\angle(\mathcal{L}, \mathcal{L}') = \frac{m'n - mn'}{mm' + nn'}. \tag{18}$$

The lines AB, AC, AD have slopes $-\frac{1}{ab}, -\frac{1}{ac}, -\frac{1}{ad}$. Let D' be the point that is isogonal to the point D with respect to the triangle ABC and let k be the slope of AD'. Then, $\angle(AB, AD) = \angle(AD', AC)$ and due to (18), we obtain

$$tg\angle(AB, AD) = \frac{ad - ab}{a^2bd + 1}, \quad tg\angle(AD', AC) = \frac{ack + 1}{ac - k}.$$

Out of the equality $\frac{ad-ab}{a^2bd+1} = \frac{ack+1}{ac-k}$, it follows that

$$k = \frac{a^2bc - a^2bd - a^2cd - 1}{a^3bcd + ab + ac - ad}. \tag{19}$$

We will show that the point

$$D' = \left(\frac{d - a - b - c}{abcd - 1}, \frac{abd + acd + bcd - abc}{abcd - 1}\right)$$

is the isogonal point to the point D with respect to the triangle ABC. Because of the symmetry on a, b, c, it is enough to show that the line AD' is isogonal to the line AD with respect to the lines AB and AC, i.e., that the line AD' have the slope k from (19). The points A and D' have the difference between the coordinates

$$\begin{aligned} a - \frac{d-a-b-c}{abcd-1} &= \frac{1}{abcd-1}(a^2bcd+b+c-d), \\ \frac{1}{a} - \frac{abd+acd+bcd-abc}{abcd-1} &= \frac{1}{a(abcd-1)(a^2bc-a^2bd-a^2cd-1)}, \end{aligned}$$

so the line AD' has the slope k in (19). The point D' can be rewritten as

$$D' = \left(\frac{2d-s}{p-1}, \frac{r-2abc}{p-1}\right). \tag{20}$$

In the same way, we can obtain the points A', B', C' isogonal to the points A, B, C with respect to the triangles BCD, ACD, ABD, respectively. The centroid of these four points is the point

$$G' = \left(-\frac{s}{2(p-1)}, \frac{r}{2(p-1)}\right). \tag{21}$$

The point D' from (20) and its analogous point C' have the midpoint $(-\frac{a+b}{p-1}, \frac{a+b}{p-1}cd)$ that lies on the line AB from (3). The line $C'D'$ has the slope ab; hence, it is perpendicular to the line AB. This means that AB is the bisector of the line segment $C'D'$. Similarly, the same is valid for the analogous elements of the quadrangles $ABCD$ and $A'B'C'D'$. Because of this, the sides AB, AC, AD, BC, BD, CD of the quadrangle $ABCD$ are bisectors of the sides $C'D', B'D', B'C', A'D', A'C', A'B'$, respectively. Out of the earlier facts, it follows that the points A, B, C, D are the centers of the circles $B'C'D'$, $A'C'D'$, $A'B'D'$, $A'B'C'$ that can be directly proved analytically, because for the distance of the point D' from the point $A = (a, \frac{1}{a})$, we obtain

$$a^2(p-1)^2AD'^2 = a^2(a^2bcd+b+c-d)^2 + [a^2(nc-bd-cd)-1]^2 = (a^2b^2+1)(a^2c^2+1)(a^2d^2+1),$$

so by analogy, we conclude that $AD' = AC' = AB'$. We proved the statement found in [2,12]:

Theorem 11. *The points A, B, C, D are the centers of the circles $B'C'D'$, $A'C'D'$, $A'B'D'$, $A'B'C'$.*

This is in addition to the statement found in [5]:

Theorem 12. *The points A', B', C', D' are isogonal to the points A, B, C, D with respect to the triangles BCD, ACD, ABD, ABC if and only if the points A, B, C, D are the centers of the circles $B'C'D', A'C'D', A'B'D', A'B'C'$.*

This means that the role of the quadrangle $ABCD$ for the quadrangle $A'B'C'D'$ is the same as the role of the quadrangle $O_aO_bO_cO_d$ for the quadrangle $ABCD$. However, the points A', B', C', D' are isogonal to the points A, B, C, D with respect to the triangles BCD, ACD, ABD, ABC. Therefore, the following theorem is proved:

Theorem 13. *The points A, B, C, D are isogonal to the points O_a, O_b, O_c, O_d with respect to the triangles $O_bO_cO_d$, $O_aO_cO_d$, $O_aO_bO_d$, $O_aO_bO_c$.*

It is also stated in [2,4].

For the point O_d from (17), the following equalities are valid

$$\begin{aligned} x-\frac{s}{2} &= \frac{1}{2}\Big(\frac{1}{abc}-d\Big)=-\frac{p-1}{2abc} \\ y-\frac{r}{2p} &= y-\frac{1}{2}\Big(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\Big)=\frac{1}{2}\Big(abc-\frac{1}{d}\Big)=\frac{p-1}{2d}, \\ \Big(x-\frac{s}{2}\Big)\Big(y-\frac{r}{2p}\Big) &= -\frac{1}{4p}(p-1)^2. \end{aligned} \tag{22}$$

Hence, this point, as well as points O_b, O_c, O_d are incident to the rectangular hyperbola $\mathcal{H}_0$ with the Equation (22) and the center $O' = (\frac{s}{2}, \frac{r}{2p})$. Due to that, O' is the center of the quadrangle $O_aO_bO_cO_d$ and the anticenter to $ABCD$. So, the following theorem is valid:

Theorem 14. *The center O' of the quadrangle $O_aO_bO_cO_d$ is the anticenter of the quadrangle $ABCD$. The center O of this quadrangle is the anticenter of $A'B'C'D'$.*

This theorem is our original result. Its visualization is given in Figure 2.

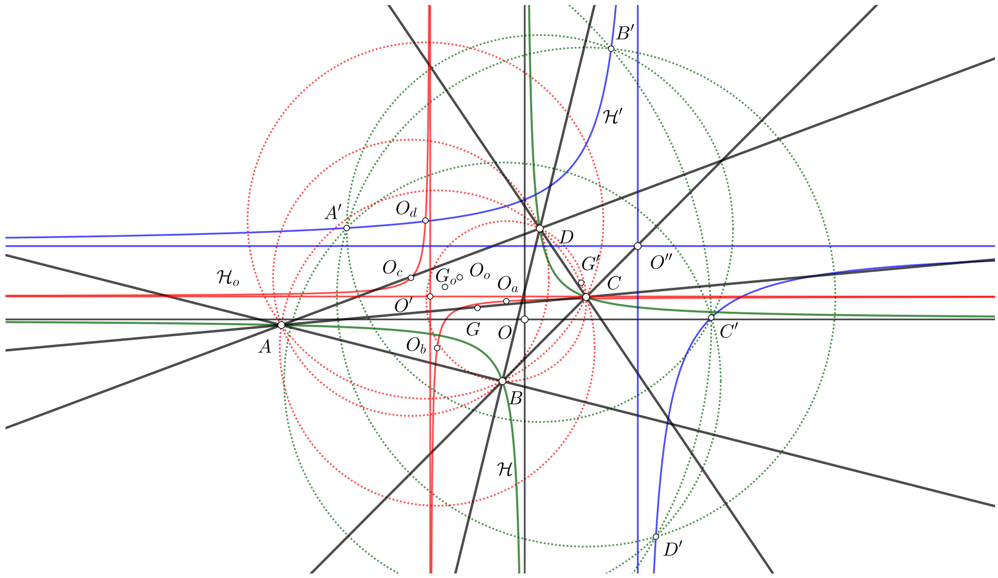

Figure 2. The visualization of Theorems 12 and 14.

The center of the quadrangle $A'B'C'D'$ is the point symmetric to the point O with respect to the centroid G' of this triangle, given by (21), so this center is the point $(-\frac{s}{p-1}, \frac{r}{p-1})$. It is easy to see that the point O_d from (17) and analogous points O_a, O_b, O_c have the centroid $G_o = (\frac{s}{8p}(3p+1), \frac{r}{8p}(p+3))$. As $O' = (\frac{s}{2}, \frac{r}{2p})$ is the center of the quadrangle $O_aO_bO_cO_d$, the anticenter is the point symmetric to the point O' with respect to the point G_o and that is the point $O_o = (\frac{s}{4p}(p+1), \frac{r}{4p}(p+1))$.

If we apply a translation for the vector $[\frac{s}{p-1}, -\frac{r}{p-1}]$ on the quadrangle $A'B'C'D'$, then, e.g., the point D' from (20) transfers to the point $D'' = (\frac{2d}{p-1}, -\frac{2abc}{p-1})$. In the same way, we can obtain the points A'', B'', C''. All the four points have the same product of the coordinates, so they are all incident to the rectangular hyperbola $\mathcal{H}''$ with the center O and with the same asymptotes as the rectangular hyperbola $\mathcal{H}$. Hence, the point O is the center of the quadrangle $A''B''C''D''$, so the point $(-\frac{s}{p-1}, \frac{r}{p-1})$ is the center of the quadrangle $A'B'C'D'$. The symmetric point to the latter point with respect to the centroid G' from (21) of the quadrangle $A'B'C'D'$ is the point O.

4. Discussion

Putting the complete quadrangle into such a coordinate system that its circumscribed hyperbola is rectangular and has the equation xy = 1 allows us to prove many known properties use the same method. We use rectangular coordinates symmetrically on four vertices and four parameters a, b, c, d which simplify the analytical computing. Thus, we came across some more quadrangles related to the referent one which allowed us to analyze published as well as original results.

That approach enabled us to extend our results in the rich geometry of a complete quadrangle, such as related to an isoptic point, which are planned to be presented in a future paper.

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