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Synchronization of Fractional Delayed Memristive Neural Networks with Jump Mismatches via Event-Based Hybrid Impulsive Controller

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Abstract: This study investigates the asymptotic synchronization in fractional memristive neural networks of the Riemann–Liouville type, considering mixed time delays and jump mismatches. Addressing the challenges associated with discrepancies in the circuit switching speed and the accuracy of the memristor, this paper introduces an enhanced model that effectively navigates these complexities. We propose two novel event-based hybrid impulsive controllers, each characterized by unique triggering conditions. Utilizing advanced techniques in inequality and hybrid impulsive control, we establish the conditions necessary for achieving synchronization through innovative Lyapunov functions. Importantly, the developed controllers are theoretically optimized to minimize control costs, an essential consideration for their practical deployment. Finally, the effectiveness of our proposed approach is demonstrated through two illustrative simulation examples.

Keywords: synchronization; fractional memristive neural networks; mixed delays; jump mismatches; event-based impulsive control

1. Introduction

Compared to continuous control techniques such as feedback [1] and adaptive control [2], impulsive control technology represents an efficient discrete control method, only modifying the system state instantaneously under specific conditions [3]. Consequently, impulsive control can save communication bandwidth and energy consumption. It has found extensive applications in time-varying delay systems [4], chaotic systems [5], and neural networks [6]. On the other hand, the event-triggered mechanism, as a control strategy that updates control information based on the system state, can effectively conserve communication resources, reduce energy consumption, and boasts high robustness and adaptability. It has been applied in fields such as neural networks [7], singular systems [8], affine systems [9], PDE systems [10], and multiagent networks [11]. However, isolated impulsive control can lead to frequent energy expenditures [12], and event-triggered control might result in communication congestion when implemented in multi-node neural networks [11]. Hence, many researchers have integrated the advantages of these two control strategies, proposing event-based impulsive control strategies and achieving significant results in the domain of neural networks [13–17].

The memristor, first proposed by Chua in 1971 [18] and physically realized by HP Laboratories in 2008 [19], is acknowledged as the fourth fundamental circuit element. Its compact size, low energy consumption and intrinsic memory function, and ability to emulate brain synapses accurately, make it an ideal basis for constructing artificial neural networks that closely mimic biological brain functions [20]. In leveraging these strengths, a category of neural networks based on memristors, referred to as MNNs, has emerged [21]. Fractional-order differential operators, in comparison to integer-order



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Copyright: © 2024 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https:// creativecommons.org/licenses/by/ 4.0/). ones, offer a more precise depiction of physical processes due to their non-local and memory properties, enhancing the adaptability of system processes [22]. The integration of traditional MNNs with fractional-order operators, resulting in fractional memristive neural networks (FMNNs) [23], has seen widespread application in image encryption [24], audio encryption [25], and secure communication [26]. Moreover, due to active device and amplifier switching speed limitations, neuron interactions often introduce time-varying delays, including discrete delays [27], distributed delays [28], and leakage delays [29]. To our knowledge, studies on fractional memristive neural networks that simultaneously consider these three types of time-varying delays have just commenced, limiting their practical engineering applications—a gap that this study aimed to bridge.

Synchronization in FMNNs, a key aspect of neural network dynamics, has attracted extensive academic interest [30,31]. Song and colleagues [32] crafted an adaptive continuous controller to robustly synchronize FMNNs, utilizing the continuous frequency distributed equivalent model as an analytical tool. Incorporating the reaction-diffusion phenomenon into FMNNs, Wu and associates [28] introduced both the pinning continuous feedback controller and the pinning continuous adaptive controller to ensure the asymptotic synchronization of FMNNs. Drawing upon the continuous hybrid adaptive controller, Kao et al. [29] delved into the Mittag–Leffler synchronization of FMNNs, specifically considering leakage delay. In the real-world context, driven-response FMNNs invariably suffer from imperfectly matched connection weights, termed jump mismatches [27,33,34]. Addressing this, Ding et al. [27] and Zhang et al. [34] explored lag projective synchronization, employing sliding-mode control and linear feedback control, respectively. Further, Zhang et al. [33] devised a feedback controller to elucidate the quasi-synchronization conundrum. Despite their successes, these methods demand substantial communication resources, necessitating continuous control data transmission, which limits their practical applicability for FMNNs synchronization. This underscores the need for an event-based impulsive control approach, aiming for efficient control with minimized costs.

In this investigation, we present a novel event-based hybrid impulsive controller, specifically tailored to synchronize Riemann–Liouville-type FMNNs faced with mixed delays and jump mismatches. Our key advancements can be distilled into the following highlights:

- With comprehensive consideration of discrete, distributed, and leakage delays, combined with jump mismatches, we augment the relevance of FMNNs to practical systems in industry.
- With the aim of achieving control objectives for a controlled network at a lower control cost, we unveil two innovative event-based hybrid impulsive controllers, incorporating both static and dynamic event-triggering mechanisms.
- Leveraging novel Lyapunov functions, we establish a duo of sufficient criteria, theoretically ensuring asymptotic synchronization in the aforestated FMNNs.

This research navigates the complexities of integrating mixed time delays and jump mismatches, enhancing model practicality and significantly challenging controller design. Furthermore, the inherent discontinuity of the memristor and the versatility of fractional-order operators make devising suitable trigger functions for our event-based hybrid impulsive controller particularly challenging, which is a crucial step to maintaining control effectiveness and avoiding Zeno behavior.

The structure of this paper unfolds as follows. In Section 2, we provide a detailed mathematical model of the aforementioned FMNNs. Section 3 lays out the foundational concepts and preliminaries essential for validating our proposed theories. In Section 4, we introduce the two proposed controllers along with their associated conditions, ensuring asymptotic synchronization, while circumventing the potential for Zeno behavior. Section 5 presents two illustrative simulation examples. Finally, Section 6 summarizes the essence and findings of this research.

Notations: The set of real numbers is denoted by $\mathbb{R} = (-\infty, +\infty)$, and the set of positive real numbers is $\mathbb{R}^+ = (0, +\infty)$. An *n*-dimensional Euclidean space is represented

by \mathbb{R}^n . Additionally, \mathbb{N} is the set of integers, while \mathbb{N}^+ is the set of positive integers. The well-known Dirac and Gamma functions are denoted by $\delta(\cdot)$ and $\Gamma(\cdot)$, respectively.

2. Model for FMNNs

Based on [29,34] and our previous work [28], the *n*-dimensional FMNNs with mixed delays and jump mismatches can be mathematically represented as shown below:

$$D_{R}^{q}\mu_{i}(t) = -\eta_{i}\mu_{i}(t-\tau_{1}(t)) + \sum_{j=1}^{n}\kappa_{ij}(\mu_{j}(t))F_{j}(\mu_{j}(t)) + \sum_{j=1}^{n}\xi_{ij}(\mu_{j}(t))G_{j}(\mu_{j}(t-\tau_{2}(t))) + \sum_{j=1}^{n}\rho_{ij}(\mu_{j}(t))\int_{t-\tau_{3}(t)}^{t}H_{j}(\mu_{j}(s))ds, \quad q \in (0,1)$$
(1)

where $i, j = 1, 2, \dots, n$. The state of the *i*th neuron associated with time t (t > 0) could be denoted as $\mu_i(t)$, and its self-feedback parameter is $\eta_i \in \mathbb{R}^+$. It is worth mentioning that the value of the memristor is dependent on the system's state. Thus, the memeristive connection weights $\kappa_{ij}(\cdot)$, $\xi_{ij}(\cdot)$, and $\rho_{ij}(\cdot)$ can be described as follows:

$$\begin{split} \kappa_{ij}(\mu_{j}(t)) &= \begin{cases} \kappa_{ij}, & |\mu_{j}(t)| > \mathcal{W}_{j} \\ \dot{\kappa}_{ij}, & |\mu_{j}(t)| \le \mathcal{W}_{j} \end{cases}, \quad \xi_{ij}(\mu_{j}(t)) = \begin{cases} \xi_{ij}, & |\mu_{j}(t)| > \mathcal{W}_{j} \\ \xi_{ij}, & |\mu_{j}(t)| \le \mathcal{W}_{j} \end{cases}, \\ \rho_{ij}(\mu_{j}(t)) &= \begin{cases} \dot{\rho}_{ij}, & |\mu_{j}(t)| > \mathcal{W}_{j} \\ \dot{\rho}_{ij}, & |\mu_{j}(t)| \le \mathcal{W}_{j} \end{cases}, \end{split}$$

where $\kappa_{ij}, \kappa_{ij}, \xi_{ij}, \hat{\xi}_{ij}, \hat{\rho}_{ij}, \hat{\rho}_{ij} \in \mathbb{R}$, and switching jumps satisfy $\mathcal{W}_j \in \mathbb{R}^+$. The activation functions $F_j(\cdot), G_j(\cdot)$, and $H_j(\cdot)$ satisfy some conditions that are shown in Assumption 1. $\tau_1(t), \tau_2(t)$, and $\tau_3(t)$ are time-varying leakage delays, discrete delays, and distributed delays, respectively, in which $\tau_m(t) \in (0, \tau_m), \tau_m \in \mathbb{R}^+, \dot{\tau}_m(t) \in (0, \mathfrak{h}_m), \mathfrak{h}_m \in (0, 1)$, and m = 1, 2, 3. The following are the initial criteria for FMNNs (1):

$$D_R^{q-1}\mu_i(s) = \phi_i(s), \ s \in [-\tau_{max}, 0], \ \tau_{max} = \max\{\tau_1, \tau_2, \tau_3\},\tag{2}$$

where $\phi_i(s)$ is bounded and continuous on $[-\tau_{max}, 0]$.

Remark 1. Memristive neural networks (MNNs) inherently face time delays due to circuit limitations, which can compromise their performance. Previous research [27–29,35] has tended to address discrete, distributed, or leakage delays either singly or in pairs, limiting holistic comprehension and application. Our work expands the conventional FMNN model to include these mixed delays, significantly improving its applicability and accuracy in real-world scenarios.

3. Preliminaries

Definition 1 ([35]). *The Riemann–Liouville fractional integral and derivative are respectively defined as follows:*

$$D_R^{-q}\mu(t) = \frac{1}{\Gamma(q)} \int_{t_0}^t (t-s)^{(q-1)}\mu(s) ds,$$

$$D_R^{q}\mu(t) = \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_{t_0}^t (t-s)^{(n-q-1)}\mu(s) ds,$$

where $q \in (n - 1, n)$ for $n \in \mathbb{N}^+$, and t_0 is the initial moment of the function $\mu(\cdot)$.

Property 1 ([35]). *For a function* $\mu(t)$ *where* $p \in (0, +\infty)$ *,* $q \in (0, p)$ *and* $c \in \mathbb{R}$ *, the following inequalities hold:*

$$D_R^q(D_R^{-p}\mu(t)) = D_R^{q-p}\mu(t), \quad D_R^q(c\mu(t)) = cD_R^q\mu(t).$$

Lemma 1 ([35]). In assuming the existence of a differentiable function vector $\mu(t) : \mathbb{R}^n \to \mathbb{R}^n$, the following inequality holds:

$$D^q_R\Big(\mu^T(t)\mu(t)\Big) \leq 2\mu^T(t)D^q_R\mu(t), \forall q \in (0,1).$$

Lemma 2 ([35]). The inequality $2\epsilon_1^T \epsilon_2 \leq \iota \epsilon_1^T \epsilon_1 + \frac{1}{\iota} \epsilon_2^T \epsilon_2$ always holds $\forall \epsilon_1, \epsilon_2 \in \mathbb{R}^n$ and $\iota \in \mathbb{R}^+$.

Lemma 3 ([28]). The integration of the vector function $\epsilon(t)$: $[\varsigma_1, \varsigma_2] \to \mathbb{R}^n$ is well defined as

$$\left[\int_{\zeta_1}^{\varsigma_2} \epsilon(s) ds\right]^T \left[\int_{\zeta_1}^{\varsigma_2} \epsilon(s) ds\right] \le (\varsigma_2 - \varsigma_1) \left[\int_{\varsigma_1}^{\varsigma_2} \epsilon^T(s) \epsilon(s) ds\right].$$

where $\zeta_1, \zeta_1 \in \mathbb{R}$ and $\zeta_1 < \zeta_2$.

Lemma 4 ([36] Fractional Barbalat's lemma). *The vector function, defined as* $\mu(t) : \mathbb{R}^n \to \mathbb{R}^n$, will converge to 0 if $\int_{t_0}^t \mu(s) ds$ and $D_R^q \mu(t)$ ($q \in (0,1)$) both have a finite limit when $t \to +\infty$.

Assumption 1. The neural activation functions satisfy $F_j(0) = 0$, $G_j(0) = 0$, $H_j(0) = 0$, and the following inequalities for the *j*th neuron:

$$\begin{aligned} |F_j(\epsilon_1) - F_j(\epsilon_2)| &\leq \mathcal{F}_j|\epsilon_1 - \epsilon_2|, \\ |G_j(\epsilon_1) - G_j(\epsilon_2)| &\leq \mathcal{G}_j|\epsilon_1 - \epsilon_2|, \\ |H_j(\epsilon_1) - H_j(\epsilon_2)| &\leq \mathcal{H}_j|\epsilon_1 - \epsilon_2|. \end{aligned}$$

where $\mathcal{F}_i, \mathcal{G}_i, \mathcal{H}_i \in \mathbb{R}^+$.

Lemma 5 ([34]). Based on Assumption 1, we obtain

1. $|\kappa_{ij}^{*}(\nu_{j}(t))F_{j}(\nu_{j}(t)) - \kappa_{ij}(\mu_{j}(t))F_{j}(\mu_{j}(t))| \leq \check{\kappa}_{ij}\mathcal{F}_{j}|\zeta_{j}(t)| + \mathcal{T}_{j}|\check{\kappa}_{ij} - \check{\kappa}_{ij}|,$ $2. |\zeta_{ij}^{*}(\nu_{j}(t))G_{j}(\nu_{j}(t-\tau_{2}(t))) - \xi_{ij}(\mu_{j}(t))G_{j}(\mu_{j}(t-\tau_{2}(t)))| \leq \check{\xi}_{ij}\mathcal{G}_{j}|\zeta_{j}(t-\tau_{2}(t))| + \mathcal{T}_{j}|\xi_{ij} - \check{\xi}_{ij}|,$

where $\check{\kappa}_{ij} = \max\{|\check{\kappa}_{ij}|, |\check{\kappa}_{ij}|\}, \check{\xi}_{ij} = \max\{|\check{\xi}_{ij}|, |\check{\xi}_{ij}|\}, and \mathcal{T}_j = \max\{\mathcal{W}_j, \mathcal{V}_j\}.$

Lemma 6. If Assumption 1 is satisfied, the following inequality will hold:

$$\begin{aligned} \left| \rho_{ij}^*(\nu_j(t)) \int_{t-\tau_3(t)}^t H_j(\nu_j(s)) \mathrm{d}s - \rho_{ij}(\mu_j(t)) \int_{t-\tau_3(t)}^t H_j(\mu_j(s)) \mathrm{d}s \right| \\ &\leq \check{\rho}_{ij} \mathcal{H}_j \int_{t-\tau_3(t)}^t |\zeta_j(s)| \mathrm{d}s + \tau_3 \mathcal{T}_j |\dot{\rho}_{ij} - \dot{\rho}_{ij}|, \end{aligned}$$

where $\check{\rho}_{ij} = \max\{|\acute{\rho}_{ij}|, |\grave{\rho}_{ij}|\}.$

Proof. If $|\mu_i| \leq W_i$, one has:

$$\begin{aligned} \left| \rho_{ij}^{*}(\nu_{j}(t)) \int_{t-\tau_{3}(t)}^{t} H_{j}(\nu_{j}(s)) ds - \rho_{ij}(\mu_{j}(t)) \int_{t-\tau_{3}(t)}^{t} H_{j}(\mu_{j}(s)) ds \right| \\ &\leq \left| \rho_{ij}(\nu_{j}(t)) \int_{t-\tau_{3}(t)}^{t} H_{j}(\nu_{j}(s)) - H_{j}(\mu_{j}(s)) ds \right| \\ &+ \left| \left[\rho_{ij}(\nu_{j}(t)) - \rho_{ij}(\mu_{j}(t)) \right] \int_{t-\tau_{3}(t)}^{t} H_{j}(\mu_{j}(s)) ds \right| \end{aligned} \tag{3}$$

$$&\leq \check{\rho}_{ij}\mathcal{H}_{j} \int_{t-\tau_{3}(t)}^{t} \left| H_{j}(\nu_{j}(s)) - H_{j}(\mu_{j}(s)) \right| ds + \left| \rho_{ij} - \rho_{ij} \right| \int_{t-\tau_{3}(t)}^{t} \left| H_{j}(\mu_{j}(s)) \right| ds \\ &\leq \check{\rho}_{ij}\mathcal{H}_{j} \int_{t-\tau_{3}(t)}^{t} \left| \zeta_{j}(s) \right| ds + \left| \rho_{ij} - \rho_{ij} \right| \tau_{3}\mathcal{H}_{j} \mathcal{W}_{j}. \end{aligned}$$

If $|\mu_i| > W_i$ and $|\nu_i| > V_i$, we obtain:

$$\begin{aligned} \left| \rho_{ij}^* (\nu_j(t)) \int_{t-\tau_3(t)}^t H_j(\nu_j(s)) \mathrm{d}s - \rho_{ij}(\mu_j(t)) \int_{t-\tau_3(t)}^t H_j(\mu_j(s)) \mathrm{d}s \right| \\ &\leq \check{\rho}_{ij} \int_{t-\tau_3(t)}^t |H_j(\nu_j(s)) - H_j(\mu_j(s))| \mathrm{d}s \\ &\leq \check{\rho}_{ij} \mathcal{H}_j \int_{t-\tau_3(t)}^t |\zeta_j(s)| \mathrm{d}s. \end{aligned}$$

$$(4)$$

If $|\mu_j| > W_j$ and $|\nu_j| \le V_j$, we have:

$$\begin{aligned} \left| \rho_{ij}^{*}(\nu_{j}(t)) \int_{t-\tau_{3}(t)}^{t} H_{j}(\nu_{j}(s)) ds - \rho_{ij}(\mu_{j}(t)) \int_{t-\tau_{3}(t)}^{t} H_{j}(\mu_{j}(s)) ds \right| \\ &\leq \left| \rho_{ij}(\mu_{j}(t)) \int_{t-\tau_{3}(t)}^{t} H_{j}(\nu_{j}(s)) - H_{j}(\mu_{j}(s)) ds \right| \\ &+ \left| \left[\rho_{ij}(\nu_{j}(t)) - \rho_{ij}(\mu_{j}(t)) \right] \int_{t-\tau_{3}(t)}^{t} H_{j}(\nu_{j}(s)) ds \right| \end{aligned}$$
(5)
$$&\leq \check{\rho}_{ij}\mathcal{H}_{j} \int_{t-\tau_{3}(t)}^{t} \left| H_{j}(\nu_{j}(s)) - H_{j}(\mu_{j}(s)) \right| ds + \left| \rho_{ij} - \rho_{ij} \right| \int_{t-\tau_{3}(t)}^{t} \left| H_{j}(\nu_{j}(s)) \right| ds \\ &\leq \check{\rho}_{ij}\mathcal{H}_{j} \int_{t-\tau_{3}(t)}^{t} \left| \zeta_{j}(s) \right| ds + \left| \rho_{ij} - \rho_{ij} \right| \tau_{3}\mathcal{H}_{j}\mathcal{V}_{j}. \end{aligned}$$

From (3)–(5), Lemma 6 always holds. The proof is completed. \Box

4. Main Results

In this section, we will explore the asymptotic synchronization problem of the FMNNs and describe the corresponding response system for (1) as follows:

$$D_{R}^{q}\nu_{i}(t) = -\eta_{i}\nu_{i}(t-\tau_{1}(t)) + \sum_{j=1}^{n}\kappa_{ij}^{*}(\nu_{j}(t))F_{j}(\nu_{j}(t)) + \sum_{j=1}^{n}\xi_{ij}^{*}(\nu_{j}(t))G_{j}(\nu_{j}(t-\tau_{2}(t))) + \sum_{j=1}^{n}\rho_{ij}^{*}(\nu_{j}(t))\int_{t-\tau_{3}(t)}^{t}H_{j}(\nu_{j}(s))ds + \psi_{i}(t),$$
(6)

where $v_i(t)$ is represented as a neuron state, and $\psi_i(t)$ is the novel controller to be designed. And $\kappa_{ii}^*(\cdot)$, $\xi_{ii}^*(\cdot)$, and $\rho_{ii}^*(\cdot)$ are described as follows:

$$\begin{split} \kappa_{ij}^*(\nu_j(t)) &= \begin{cases} \kappa_{ij}, & |\nu_j(t)| > \mathcal{V}_j \\ \kappa_{ij}, & |\nu_j(t)| \le \mathcal{V}_j \end{cases}, \quad \xi_{ij}^*(\nu_j(t)) = \begin{cases} \xi_{ij}, & |\mu_j(t)| > \mathcal{V}_j \\ \xi_{ij}, & |\mu_j(t)| \le \mathcal{V}_j \end{cases}, \\ \rho_{ij}^*(\nu_j(t)) &= \begin{cases} \rho_{ij}, & |\mu_j(t)| > \mathcal{V}_j \\ \rho_{ij}, & |\mu_j(t)| \le \mathcal{V}_j \end{cases}, \end{split}$$

where switching jumps satisfy $\mathcal{V}_i \in \mathbb{R}^+$. Just like (2), the initial criteria for FMNNs (6) are

$$D_R^{q-1}\nu(s) = \varphi_i(s), \ s \in [-\tau_{max}, 0], \ \tau_{max} = \max\{\tau_1, \tau_2, \tau_3\},\tag{7}$$

where $\varphi_i(s)$ is bounded and continuous on $[-\tau_{max}, 0]$.

Remark 2. The FMNNs, as outlined by Equations (1) and (6), are state-dependent differential systems where the connection weights are determined by the states of the neurons. Given that disturbances, whether internal or external, can lead to variations in connection weights between driven and response systems, addressing jump mismatches becomes essential.

Let $\zeta_i(t) = \nu_i(t) - \mu_i(t)$ denote the synchronization error. Therefore, according to (1) and (6), we know that the synchronization error system is

$$D_{R}^{q}\zeta_{i}(t) = -\eta_{i}\zeta_{i}(t - \tau_{1}(t)) + \sum_{j=1}^{n} \left[\kappa_{ij}^{*}(\nu_{j}(t))F_{j}(\nu_{j}(t)) - \kappa_{ij}(\mu_{j}(t))F_{j}(\mu_{j}(t))\right] \\ + \sum_{j=1}^{n} \left[\xi_{ij}^{*}(\nu_{j}(t))G_{j}(\nu_{j}(t - \tau_{2}(t))) - \xi_{ij}(\mu_{j}(t))G_{j}(\mu_{j}(t - \tau_{2}(t)))\right] \\ + \sum_{j=1}^{n} \left[\rho_{ij}^{*}(\nu_{j}(t))\int_{t-\tau_{3}(t)}^{t}H_{j}(\nu_{j}(s))ds - \rho_{ij}(\mu_{j}(t))\int_{t-\tau_{3}(t)}^{t}H_{j}(\mu_{j}(s))ds\right] + \psi_{i}(t).$$
(8)

The innovative event-based impulsive controller $\psi_i(t)$ is designed as follows:

$$\psi_i(t) = -\mathfrak{p}_{1i}\zeta_i(t_{k-1}) - \mathfrak{p}_{2i}\operatorname{sgn}\left(\zeta_i(t_{k-1})\right) + \sum_{k=1}^{+\infty} \left[\alpha_{ik}(\zeta(t)) - \zeta_i(t)\right]\delta(t-t_k),$$
(9)

where $k \in \mathbb{N}$ represents the *k*th impulsive instant. The feedback control gains are $\mathfrak{p}_{1i} \in \mathbb{R}^+$ and $\mathfrak{p}_{2i} \in \mathbb{R}$, respectively. The impulsive moment sequence $\{t_k\}$ is a monotonically increasing series and satisfies $t_k \to \infty$ as $k \to +\infty$. For the purpose of subsequent proofs, this paper supposes that the neuron's state is right-continuous in an impulsive instant, i.e., it satisfies the conditions $\zeta_i(t_k^-) = \zeta_i(t_k)$ and $\zeta(t_k^+) = \lim_{t \to t_k^+} \zeta_i(t)$. Instead of assuming the impulsive strength as the constant, we consider it the function $\alpha_{ik}(\zeta(t))$ satisfying Assumption 2.

Remark 3. The controller introduced herein was designed to efficiently achieve the control objective, using minimal resources and exhibiting resilience to unforeseen perturbations. To this end, a static event trigger strategy is detailed in Section 4.1, which is later refined to a dynamic strategy in Section 4.2 for heightened adaptability.

According to the controller (9), (8) could be translated as

$$\begin{cases} D_{R}^{q}\zeta_{i}(t) = -\eta_{i}\zeta_{i}(t-\tau_{1}(t)) + \sum_{j=1}^{n} \left[\kappa_{ij}^{*}(\nu_{j}(t))F_{j}(\nu_{j}(t)) - \kappa_{ij}(\mu_{j}(t))F_{j}(\mu_{j}(t))\right] \\ + \sum_{j=1}^{n} \left[\zeta_{ij}^{*}(\nu_{j}(t))G_{j}(\nu_{j}(t-\tau_{2}(t))) - \zeta_{ij}(\mu_{j}(t))G_{j}(\mu_{j}(t-\tau_{2}(t)))\right] \\ + \sum_{j=1}^{n} \rho_{ij}^{*}\left[(\nu_{j}(t))\int_{t-\tau_{3}(t)}^{t}H_{j}(\nu_{j}(s))ds - \rho_{ij}(\mu_{j}(t))\int_{t-\tau_{3}(t)}^{t}H_{j}(\mu_{j}(s))ds\right] \\ -\mathfrak{p}_{1i}\zeta_{i}(t_{k-1}) - \mathfrak{p}_{2i}\operatorname{sgn}\left(\zeta_{i}(t_{k-1})\right), \qquad t \neq t_{k}, \\ \zeta_{i}(t_{k}^{+}) = \alpha_{ik}(\zeta(t)), \qquad t = t_{k}. \end{cases}$$
(10)

The specific control process is depicted in Figure 1. The value of the error FMNNs $(\zeta(t))$ is obtained by calculating the difference between the states of the neurons in the response FMNNs $(\nu(t))$ and the driven FMNNs $(\mu(t))$ at the current instant. The signal receiver in the controller collects the value of the error system neurons and decides whether to update the value of the signal exporter based on the trigger condition. The response FMNNs update the state of their own neurons according to the output signals given by the controller $(\psi(t))$ in order to facilitate asymptotic synchronization with the driven FMNNs.

Assumption 2. The impulsive strength function $\alpha_{ik}(\zeta(t))$ satisfies

$$\left|\alpha_{ik}(\epsilon_1(t),\epsilon_2(t),\cdots,\epsilon_n(t))\right|^{\varrho}\leq \sum_{j=1}^n \alpha_{ij}^k |\mu_j(t)|^{\varrho},$$

where $\alpha_{ij}^k \in (0, +\infty)$, $\varrho \in \mathbb{N}^+$, and $k \in \mathbb{N}$.



Figure 1. Schematic block diagram for synchronization of FMNNs.

Remark 4. Within numerous impulsive control frameworks, the impulsive strength is often assumed to be constant, expressed as $\mu_i(t_k^+) = r_{ik}\mu_i(t_k)$, where r_{ik} is a constant. It is evident that Assumption 2 is satisfied when $\alpha_{ii}^k = r_{ik}$ and $\alpha_{ij}^k = 0$ for $i \neq j$. Consequently, the controller formulated in this study offers a more encompassing and versatile approach.

4.1. Static Event Trigger Strategy

Let the measurement error function be

$$\varepsilon_i(t) = \zeta_i(t_{k-1}) - \zeta_i(t), \ t \in [t_{k-1}, t_k).$$
(11)

Then, the trigger function is defined as follows:

$$\omega(t) = \varepsilon_{max}(t) - \frac{\iota \Xi_{min} \zeta_{min}(t_{k-1}) + \varsigma \Theta_{min}}{\mathfrak{p}_{1max} + \iota \Xi_{max}},$$
(12)

where $\iota, \varsigma \in \mathbb{R}$, $\varepsilon_{max}(t) = \max\{|\varepsilon_i(t)|\}, \zeta_{min}(t_{k-1}) = \min\{|\zeta_i(t_{k-1})|\}, \mathfrak{p}_{1max} = \max\{\mathfrak{p}_{1i}\},$ $\Xi_{max} = \max{\{\Xi_i\}, \Xi_{min} = \min{\{\Xi_i\}, \Theta_{min} = \min{\{\Theta_i\}} \text{ and }$

$$\begin{split} \Xi_{i} &= -\frac{1}{2} \sum_{j=1}^{n} \frac{\eta_{i}}{1 - \mathfrak{h}_{1}} + \breve{\kappa}_{ij} + \breve{\kappa}_{ji} \mathcal{F}_{i}^{2} + \breve{\xi}_{ij} + \frac{\breve{\xi}_{ji}}{1 - \mathfrak{h}_{2}} \mathcal{G}_{i}^{2} + \breve{\rho}_{ij} + \frac{\breve{\rho}_{ji}}{1 - \mathfrak{h}_{3}} \mathcal{H}_{i}^{2} \tau_{3} + \mathfrak{p}_{1i},\\ \Theta_{i} &= -\sum_{j=1}^{n} |\kappa_{ij} - \breve{\kappa}_{ij}| \mathcal{T}_{j} + |\breve{\xi}_{ij} - \breve{\xi}_{ij}| \mathcal{T}_{j} + |\rho_{ij} - \rho_{ij}| \mathcal{T}_{j} \tau_{3} + \bar{\mathfrak{p}}_{2i}, \end{split}$$

where $\begin{cases} \mathfrak{p}_{2i} \geq \bar{\mathfrak{p}}_{2i}, & \frac{\operatorname{sgn}(\zeta_i(t_{k-1}))}{\operatorname{sgn}(\zeta_i(t))} > 0, \\ \mathfrak{p}_{2i} < -\bar{\mathfrak{p}}_{2i}, & others, \end{cases}$. Thus, the static event trigger condition is

$$t_k = \inf\{t \in (t_{k-1}, +\infty) | \omega(t) > 0\}.$$
(13)

Theorem 1. With the assistance of the proposed controller (9) with static trigger condition (13), asymptotic synchronization is achieved in FMNNs (1) and (6) if Assumptions 1 and 2 and the following inequalities hold:

- $\begin{array}{ll} 1. & 0 < \sum_{j=1}^{n} \alpha_{ij}^{k} < 1; \\ 2. & \Xi_{i}, \Theta_{i} \geq 0; \end{array}$
- 3. $\iota, \varsigma, \sum_{i=1}^{n} \alpha_{ii}^{k} \in (0, 1).$

Proof. Pick an appropriate Lyapunov function:

$$V(t) = V_1(t) + V_2(t) + V_3(t) + V_4(t),$$
(14)

where

$$\begin{split} V_1(t) &= \frac{1}{2} \sum_{i=1}^n D_R^{q-1} \zeta_i^2(t), \\ V_2(t) &= (2 - 2\mathfrak{h}_2)^{-1} \sum_{i=1}^n \sum_{j=1}^n \eta_i \int_{t-\tau_1(t)}^t \zeta_i^2(s) \mathrm{d}s, \\ V_3(t) &= (2 - 2\mathfrak{h}_2)^{-1} \sum_{i=1}^n \mathcal{G}_i^2 \sum_{j=1}^n \check{\xi}_{ij} \int_{t-\tau_1(t)}^t \zeta_i^2(s) \mathrm{d}s, \\ V_4(t) &= \tau_3 (2 - 2\mathfrak{h}_3)^{-1} \sum_{i=1}^n \mathcal{H}_i^2 \sum_{j=1}^n \check{\rho}_{ij} \int_{-\tau_3(t)}^0 \int_{t+s}^t \zeta_i^2(\theta) \mathrm{d}\theta \mathrm{d}s. \end{split}$$

When $t \neq t_k$, according to Lemmas 5 and 6, the derivative of $V_1(t)$ is

$$\begin{split} \dot{V}_{1}(t) &\leq \sum_{i=1}^{n} \zeta_{i}(t) D_{R}^{q} \zeta_{i}(t) \\ &\leq \sum_{i=1}^{n} \Big\{ -\eta_{i} \zeta_{i}(t) \zeta_{i}(t-\tau_{1}(t)) + |\zeta_{i}(t)| \sum_{j=1}^{n} \big[\breve{\kappa}_{ij} \mathcal{F}_{j} |\zeta_{j}(t)| + \mathcal{T}_{j} |\breve{\kappa}_{ij} - \breve{\kappa}_{ij}| \big] \\ &+ |\zeta_{i}(t)| \sum_{j=1}^{n} \big[\breve{\xi}_{ij} \mathcal{G}_{j} |\zeta_{j}(t-\tau_{2}(t))| + \mathcal{T}_{j} |\breve{\xi}_{ij} - \mathring{\xi}_{ij}| \big] + |\zeta_{i}(t)| \sum_{j=1}^{n} \big[\breve{\rho}_{ij} \mathcal{H}_{j} \int_{t-\tau_{3}(t)}^{t} |\zeta_{j}(s)| ds \\ &+ \tau_{3} \mathcal{T}_{j} |\dot{\rho}_{ij} - \dot{\rho}_{ij}| \big] - \mathfrak{p}_{1i} \zeta_{i}(t) \zeta_{i}(t_{k-1}) - \mathfrak{p}_{2i} \operatorname{sgn} \big(\zeta_{i}(t_{k-1}) \big) \zeta_{i}(t) \Big\}. \end{split}$$

According to Lemmas 2 and 3, one obtains

$$-\eta_i \zeta_i(t) \zeta_i(t - \tau_1(t)) \le \frac{1}{2} \eta_i \big(\zeta_i^2(t) + \zeta_i^2(t - \tau_1(t)) \big), \tag{16}$$

$$|\zeta_i(t)|\sum_{j=1}^n \breve{\kappa}_{ij}\mathcal{F}_j|\zeta_i(t)| \le \frac{1}{2}\sum_{j=1}^n \left(\breve{\kappa}_{ij} + \breve{\kappa}_{ji}\mathcal{F}_i^2\right)\zeta_i^2(t),\tag{17}$$

$$|\zeta_i(t)|\sum_{j=1}^n \check{\xi}_{ij}\mathcal{G}_j|\zeta_i(t-\tau_2(t))| \le \frac{1}{2}\sum_{j=1}^n \check{\xi}_{ij}\zeta_i^2(t) + \check{\xi}_{ji}\mathcal{G}_i^2\zeta_i^2(t-\tau_2(t)),$$
(18)

$$\begin{aligned} |\zeta_{i}(t)| \sum_{j=1}^{n} \check{\rho}_{ij} \mathcal{H}_{j} \int_{t-\tau_{3}(t)}^{t} |\zeta_{j}(s)| \mathrm{d}s &\leq \frac{1}{2} \sum_{j=1}^{n} \check{\rho}_{ij} \zeta_{i}^{2}(t) + \check{\rho}_{ji} \mathcal{H}_{i}^{2} \left(\int_{t-\tau_{3}(t)}^{t} |\zeta_{j}(s)| \mathrm{d}s\right)^{2} \\ &\leq \frac{1}{2} \sum_{j=1}^{n} \check{\rho}_{ij} \zeta_{i}^{2}(t) + \check{\rho}_{ji} \mathcal{H}_{i}^{2} \tau_{3} \int_{t-\tau_{3}(t)}^{t} \zeta_{j}^{2}(s) \mathrm{d}s. \end{aligned}$$
(19)

Combining (16)–(19) with (15), one could obtain

$$\begin{split} \dot{V}_{1}(t) &\leq \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\eta_{i} + \breve{\kappa}_{ij} + \breve{\kappa}_{ji} \mathcal{F}_{i}^{2} + \breve{\xi}_{ij} + \breve{\rho}_{ij} \right) \zeta_{i}^{2}(t) \\ &+ \sum_{i=1}^{n} \sum_{j=1}^{n} \left(|\breve{\kappa}_{ij} - \breve{\kappa}_{ij}| \mathcal{T}_{j} + |\breve{\xi}_{ij} - \breve{\xi}_{ij}| \mathcal{T}_{j} + |\rho_{ij} - \rho_{ij}| \mathcal{T}_{j} \tau_{3} \right) |\zeta_{i}(t)| \\ &+ \frac{1}{2} \sum_{i=1}^{n} \left[\eta_{i} \zeta_{i}(t - \tau_{1}(t)) + \sum_{j=1}^{n} \breve{\xi}_{ji} \mathcal{G}_{i}^{2} \zeta_{i}^{2}(t - \tau_{2}(t)) + \breve{\rho}_{ji} \mathcal{H}_{i}^{2} \tau_{3} \int_{t - \tau_{3}(t)}^{t} \zeta_{i}^{2}(s) \mathrm{d}s \right] \\ &- \sum_{i=1}^{n} \mathfrak{p}_{1i} \zeta_{i}(t) \zeta_{i}(t_{k-1}) - \sum_{i=1}^{n} \mathfrak{p}_{2i} \zeta_{i}(t) \operatorname{sgn}(\zeta_{i}(t_{k-1})). \end{split}$$

At the same time, one could obtain

$$\dot{V}_{2}(t) = (2 - 2\mathfrak{h}_{1})^{-1} \sum_{i=1}^{1} \sum_{j=1}^{n} \eta_{i} \Big(\zeta_{i}^{2}(t) - (1 - \dot{\tau}_{1}(t)) \zeta_{i}^{2}(t - \tau_{1}(t)) \Big) \\ \leq \sum_{i=1}^{n} \sum_{j=1}^{n} \Big(\frac{\eta_{i}}{2(1 - \mathfrak{h}_{1})} \zeta_{i}^{2}(t) - \frac{\eta_{i}}{2} \zeta_{i}^{2}(t - \tau_{2}(t)) \Big),$$
(21)

$$\begin{split} \dot{V}_{3}(t) &= (2 - 2\mathfrak{h}_{2})^{-1} \sum_{i=1}^{1} \mathcal{G}_{i}^{2} \sum_{j=1}^{n} \check{\xi}_{ji} \Big(\zeta_{i}^{2}(t) - \big(1 - \dot{\tau}_{2}(t)\big) \zeta_{i}^{2}(t - \tau_{2}(t)) \Big) \\ &\leq \sum_{i=1}^{n} \sum_{j=1}^{n} \Big(\frac{\check{\xi}_{ji} \mathcal{G}_{i}^{2}}{2(1 - \mathfrak{h}_{2})} \zeta_{i}^{2}(t) - \frac{\check{\xi}_{ji}}{2} \mathcal{G}_{i}^{2} \zeta_{i}^{2}(t - \tau_{2}(t)) \Big), \end{split}$$
(22)

$$\begin{aligned} \dot{V}_{4}(t) &= \tau_{3}(2-2\mathfrak{h}_{3})^{-1}\sum_{i=1}^{n}\mathcal{H}_{i}^{2}\sum_{j=1}^{n}\check{\rho}_{ij}\Big(\tau_{3}(t)\zeta_{i}^{2}(t) - (1-\dot{\tau}_{3}(t))\int_{t-\tau_{3}(t)}^{t}\zeta_{i}^{2}(s)\mathrm{d}s\Big) \\ &\leq \sum_{i=1}^{n}\sum_{j=1}^{n}\Big(\frac{\check{\rho}_{ij}\mathcal{H}_{i}^{2}\tau_{3}}{2(1-\mathfrak{h}_{3})}\zeta_{i}^{2}(t) - \frac{\check{\rho}_{ij}\mathcal{H}_{i}^{2}\tau_{3}}{2}\int_{t-\tau_{3}(t)}^{t}\zeta_{i}^{2}(s)\mathrm{d}s\Big). \end{aligned}$$
(23)

From (20) to (23) and the measurement error function (11), we could obtain

$$\dot{V}(t) \leq \sum_{i=1}^{n} \left[-(\Xi_{i} - \mathfrak{p}_{1i})\zeta_{i}^{2}(t) - (\Theta_{i} - \bar{\mathfrak{p}}_{2i})|\zeta_{i}(t)| -\mathfrak{p}_{1i}\zeta_{i}(t)\zeta_{i}(t_{k-1}) - \mathfrak{p}_{2i}\zeta_{i}(t)\operatorname{sgn}(\zeta_{i}(t_{k-1}))\right]$$

$$\leq \sum_{i=1}^{n} \left[-\Xi_{i}\zeta_{i}^{2}(t) - \Theta_{i}|\zeta_{i}(t)|\right] + \sum_{i=1}^{n} \mathfrak{p}_{1i}|\zeta_{i}(t)||\varepsilon_{i}(t)|.$$

$$(24)$$

Moreover, the following inequalities hold according to the measurement error function (11) and $\omega(t) \leq 0$:

$$\begin{aligned} \mathfrak{p}_{1i}|\varepsilon_{i}(t)| &\leq s_{1max}\varepsilon_{max}(t) \\ &\leq \iota \Xi_{min}\zeta_{min}(t_{k-1}) + \varsigma \Theta_{min} - \iota \Xi_{max}\varepsilon_{max}(t) \\ &\leq \iota \Xi_{i}|\zeta_{i}(t_{k-1})| + \varsigma \Theta_{i} - \iota \Xi_{i}|\varepsilon_{i}(t)| \\ &\leq \iota \Xi_{i}|\zeta_{i}(t)| + \varsigma \Theta_{i}. \end{aligned}$$

$$(25)$$

Putting (25) into (24), one obtains $\dot{V}(t) \leq \sum_{i=1}^{n} (\iota - 1) \Xi_i \zeta_i^2(t) + (\varsigma - 1) \Theta_i |\zeta_i(t)| < 0$. When $t = t_k$, according to Assumption 2 and Theorem 1, one obtains

$$V_1(t_k^+) = \frac{1}{2} \sum_{i=1}^n D_R^{q-1} \zeta_i^2(t_k^+) \le \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \alpha_{ij}^k D_R^{q-1} \zeta_i^2(t_k) \le V_1(t_k).$$

Thus, inequality $V(t_k^+) \leq V(t_k)$ will hold.

Let $B(t) = \sum_{i=1}^{n} \zeta_i^2(t)$ hold; then, one could easily obtain $V(t) + \frac{1}{2} \int_{\Phi} B(t) dt \le V(t_0)$. Consequently, B(t) is bounded when $t \to +\infty$. According to Equation (10), $\left(D_R^q \zeta_i(t)\right)^2$ is bounded. From Lemma 4, $\sum_{i=1} \zeta_i^2(t) \to 0$ as $t \to +\infty$. The proof is completed. \Box

Remark 5. Our research diverges from the existing literature [27–29,32,34,35] by implementing an event-based hybrid impulsive controller, as opposed to continuous or partially discrete controllers. This innovative strategy significantly reduces the need for continuous control information transmission, thereby decreasing energy and bandwidth consumption.

Definition 2 ([37]). When $\exists h \in \mathbb{R}^+$ s.t. $\inf_{k \in N} \{t_k - t_{k-1}\} > h$, the system is said to evade Zeno behavior under the effect of the proposed controller.

Theorem 2. *The FMNNs (6) could evade Zeno behavior with the effect of controller (9) under static trigger function (13) when Theorem 1 is satisfied.*

Proof. Based on Theorem 1, we could obtain that $(D_R^q \zeta_i(t))^2$ is bounded. Thus, one could simpley infer that $D_R^q \zeta_i(t)$ is also bounded, i.e.,

$$\exists P_i \in \mathbb{R}^+, \quad \text{s.t.} \ |D^q_R \zeta_i(t)| \le P_i. \tag{26}$$

According to the Riemann–Liouville fractional definition (Definition 1) and relating property (Property 1), one could obtain the following inequality:

$$\begin{split} |\varepsilon_{i}(t)| &= |\zeta_{i}(t) - \zeta_{i}(t_{k-1})| = |D_{R}^{-q} (D_{R}^{q} \zeta_{i}(t)) - D_{R}^{-q} (D_{R}^{q} \zeta_{i}(t_{k-1}))| \\ &= |(\Gamma(q))^{-1} (\int_{t_{0}}^{t} D_{R}^{q} \zeta_{i}(s)(t-s)^{q-1} ds - \int_{t_{0}}^{t_{k-1}} D_{R}^{q} \zeta_{i}(s)(t_{k-1}-s)^{q-1} ds)| \\ &\leq (\Gamma(q))^{-1} |\int_{t_{0}}^{t_{k-1}} (D_{R}^{q} \zeta_{i}(s)(t-s)^{q-1} - \zeta_{i}(s)(t_{k-1}-s)^{q-1}) ds| \\ &+ (\Gamma(q))^{-1} |\int_{t_{k-1}}^{t} D_{R}^{q} \zeta_{i}(s)(t-s)^{q-1} ds|. \end{split}$$

According to (26), one could obtain

$$\begin{aligned} \left| \varepsilon_{i}(t) \right| &\leq (\Gamma(q))^{-1} P_{i} \Big| \int_{t_{0}}^{t_{k-1}} (t-s)^{q-1} - (t_{k-1}-s)^{q-1} ds \Big| \\ &+ (\Gamma(q))^{-1} P_{i} \Big| \int_{t_{k-1}}^{t} (t-s)^{q-1} ds \Big| \\ &= (\Gamma(q+1))^{-1} P_{i} \big(|(t-t_{k-1})^{q} - (t-t_{0})^{q} + (t_{k-1}-t_{0})^{q} | + (t-t_{k-1})^{q} \big) \\ &\leq 2 (\Gamma(q+1))^{-1} P_{i} (t-t_{k-1}). \end{aligned}$$

Obviously, the inequality always holds:

$$\varepsilon_{max}(t) \le 2(\Gamma(q+1))^{-1} P_{max}(t-t_{k-1}), \quad P_{max} = \max\{P_i\}.$$
 (27)

According to trigger condition (13), one obtains

$$\ell(t_{k-1}) = \frac{\iota \Xi_{\min} \zeta_{\min}(t_{k-1}) + \varsigma \Theta_{\min}}{s_{1\max} + \iota \Xi_{\max}} \leq \varepsilon_{\max}(t).$$

Thus, the lower bound of T_{k-1} is

$$0 < \hbar < \frac{1}{2} P_{max}^{-1} \ell(t) \Gamma(q+1) < T_{k-1},$$

where $T_{k-1} = t - t_{k-1}$. Based on Definition 2, the proof is completed. \Box

4.2. Dynamic Event Trigger Strategy

To further enhance the flexibility of the controller, we defined the auxiliary dynamic function (28), which translates the static trigger condition (13) into a dynamic one:

$$\dot{\omega}(t) = -\omega(t) + \sum_{i=1}^{n} \left(\iota \Xi_i \zeta_i^2(t) + \varsigma \Theta_i |\zeta_i(t)| - \mathfrak{p}_{1i} |\varepsilon_i(t)| |\zeta_i(t)| \right), \tag{28}$$

with the initial conditions $\omega(0) > 0$. Thus, we define the dynamic trigger function as

$$v(t) = \omega(t) - \frac{\mathfrak{p}_{1min}}{(\mathfrak{p}_{1max} + \iota \Xi_{max})\zeta_{max}(t)} \varpi(t),$$
⁽²⁹⁾

where $\zeta_{max}(t) = \max{\{\zeta_i(t)\}}$, and $\mathfrak{p}_{1min} = \min{\{\mathfrak{p}_{1i}\}}$. Thus, the dynamic trigger condition is

$$t_k = \inf\{t \in (t_{k-1}, +\infty) | v(t) > 0\},\tag{30}$$

Theorem 3. The controller with dynamic trigger condition (30) will help FMNNs (1) and FMNNs (6) achieve asymptotic synchronization if Theorem 1 holds.

Proof. We construct the new Lyapunov function $\bar{V}(t) = V(t) + \omega(t)$. According to inequality (24), its derivative is

$$\dot{V}(t) \le -\omega(t) + \sum_{i=1}^{n} \left((\iota - 1) \Xi_i \zeta_i^2(t) + (\varsigma - 1) \Theta_i |\zeta_i(t)| \right), \ t \in [t_{k-1}, t_k).$$
(31)

Owing to the trigger function $v(t) \le 0$ and the measurement error function (11), the following inequality is obtained:

$$\begin{aligned} \mathfrak{p}_{1i}|\varepsilon_{i}(t)| &\leq s_{1max}\varepsilon_{max}(t) \leq \iota \Xi_{min}\zeta_{min}(t_{k-1}) + \varsigma \Theta_{min} - \iota \Xi_{max}\varepsilon_{max}(t) + \frac{s_{1min}}{\zeta_{max}(t)} \mathcal{O}(t) \\ &\leq \frac{\mathfrak{p}_{1i}}{|\zeta_{i}(t)|} \mathcal{O}(t) + \iota \Xi_{i}|\zeta_{i}(t_{k-1})| + \varsigma \Theta_{i} - \iota \Xi_{i}|\varepsilon_{i}(t)| \\ &\leq \frac{\mathfrak{p}_{1i}}{|\zeta_{i}(t)|} \mathcal{O}(t) + \iota \Xi_{i}|\zeta_{i}(t)| + \varsigma \Theta_{i}. \end{aligned}$$
(32)

Putting (32) into (28), one could obtain $\dot{\varpi}(t) \ge -(1 + \sum_{i=1}^{n} \mathfrak{p}_{1i})\varpi(t)$. If $\bar{\varpi}$ is the solution of differential equation $\dot{\bar{\varpi}}(t) = -(1 + \sum_{i=1}^{n} \mathfrak{p}_{1i})\bar{\varpi}(t)$ with initial condition $\bar{\varpi}(0) \ge 0$, then the following inequality holds:

$$\bar{\varpi}(t) = \bar{\varpi}(0)e^{-(1+\sum_{i=1}^{n}\mathfrak{p}_{1i})t} \ge 0.$$

Based on the comparison principle, the following inequality holds:

$$\omega(t) \ge \bar{\omega}(t) \ge 0. \tag{33}$$

Putting (33) into (31), the inequality $\dot{V}(t) < 0$ holds. When $t = t_k$, one could obtain the following inequality:

$$\bar{V}(t_k^+) = V(t_k^+) + \mathcal{O}(t_k^+) \le V(t) + \mathcal{O}(t) \le \bar{V}(t).$$

$$(34)$$

The proof is completed. \Box

Remark 6. The controller is activated by a static event trigger condition, given by Equation (13), when $\omega(t)$ exceeds zero. This suggests that the difference between the error system value $\zeta_i(t_{k-1})$ at the last triggering instant and $\zeta_i(t)$ at the current time exceeds a fixed constant $(\iota \Xi_{min} \zeta_{min}(t_{k-1}) + \iota \Xi_{min} \zeta_{min}(t_{k-1}))$

 $\varsigma \Theta_{min}$ / ($\mathfrak{p}1max + \iota \Xi max$). To enhance adaptability and reduce trigger occurrences, we incorporate an auxiliary dynamic function $\mathfrak{O}(t)$ to adjust the triggering criterion.

Theorem 4. *Similar to in Theorem 2, FMNNs (6) could also evade Zeno behavior with controller* (9) *under dynamic trigger function (30) when Theorem 3 is satisfied.*

Proof. When $t \in [t_{k-1}, t_k)$, one obtains the following inequality based on the dynamic trigger function $v(t) \le 0$, $t \in [t_{k-1}, t_k)$:

$$\bar{\ell}(t) = \frac{\iota \Xi_{\min} \zeta_{\min}(t_{k-1}) + \varsigma \Theta_{\min}}{\mathfrak{p}_{1max} + \iota \Xi_{max}} + \frac{\mathfrak{p}_{1min}}{(\mathfrak{p}_{1max} + \iota \Xi_{max}) \zeta_{max}(t)} \mathcal{O}(t) \le \varepsilon_{max}(t).$$

Owing to (27), the lower bound of T_{k-1} is

$$0 < \bar{\hbar} = \frac{1}{2} P_{max}^{-1} \bar{\ell}(t) \Gamma(q+1) < T_{k-1},$$

where $T_{k-1} = t - t_{k-1}$. The proof is completed. \Box

Remark 7. It can be observed that the control design in this paper includes many fixed parameters, such as Θ , Ξ , and so on. However, in the practical application of networks, excessive parameter designs often increase the engineering complexity. Therefore, reducing the total number of parameters in such controllers will be a problem worthy of careful consideration.

5. Numerical Example

This section presents two examples to illustrate the effectiveness of the proposed controller.

Example 1. *Initially, we examine the static event trigger strategy within three-neuron FMNNs with mixed delays. The model can be represented as follows:*

$$D_{R}^{q}\mu_{i}(t) = -\eta_{i}\mu_{i}(t-\tau_{1}(t)) + \sum_{j=1}^{3}\kappa_{ij}(\mu_{j}(t))F_{j}(\mu_{j}(t)) + \sum_{j=1}^{3}\xi_{ij}(\mu_{j}(t))G_{j}(\mu_{j}(t-\tau_{2}(t))) + \sum_{j=1}^{3}\rho_{ij}(\mu_{j}(t))\int_{t-\tau_{3}(t)}^{t}H_{j}(\mu_{j}(s))ds,$$
(35)

with the parameters set as q = 0.8 and $\eta_1 = \eta_2 = \eta_3 = 1$. The leakage, discrete, and distributed delays are $\tau_1(t) = \frac{e^t}{1+e^t}$, $\tau_2(t) = 0.6\tau_1(t)$, and $\tau_3(t) = 0.2\tau_1(t)$, respectively. Therefore, $\tau_1 = 1$, $\tau_2 = 0.6$, and $\tau_3 = 0.2$ and $\mathfrak{h}_1 = 0.25$, $\mathfrak{h}_2 = 0.15$, and $\mathfrak{h}_3 = 0.05$. The activation functions were chosen as $F_j(\mu_i(t)) = \tanh(\mu_j(t))$, $G_j(\mu_i(t - \tau_2(t))) = \sin(\mu_j(t - \tau_2(t)))$, and $H_j(\mu_j(t)) = 0.3 \sin(\mu_j(t))$, which implies that $\mathcal{F}_j = \mathcal{G}_j = 1$ and $\mathcal{H}_j = 0.3$. The connection weights are

$$\begin{aligned} & (\kappa_{ij}) = \begin{bmatrix} -0.5 & 0.3 & -0.5 \\ -0.9 & 0.2 & -0.1 \\ 0.3 & -0.1 & -0.3 \end{bmatrix}, \quad (\kappa_{ij}) = \begin{bmatrix} 0.3 & 0.6 & 0.45 \\ -0.7 & 0.3 & 0.3 \\ -0.25 & 0.25 & 0.3 \end{bmatrix}, \\ & (\xi_{ij}) = \begin{bmatrix} -0.8 & 0.45 & -0.8 \\ -0.9 & 0.2 & -0.1 \\ -0.35 & -0.2 & -0.3 \end{bmatrix}, \quad (\xi_{ij}) = \begin{bmatrix} -0.3 & 0.3 & 0.1 \\ -0.8 & 0.3 & 0.2 \\ -0.45 & 0.3 & 0.1 \end{bmatrix}, \\ & (\rho_{ij}) = \begin{bmatrix} -0.4 & 0.2 & -0.6 \\ -0.6 & -0.2 & -0.1 \\ -0.25 & -0.15 & -0.3 \end{bmatrix}, \quad (\rho_{ij}) = \begin{bmatrix} 0.2 & -0.5 & 0.3 \\ 0.4 & 0.3 & 0.2 \\ 0 & 0.2 & 0.1 \end{bmatrix}, \end{aligned}$$

and $W_j = 2$. The initial conditions of system (35) are $D_R^{q-1}\mu_i(s) = (1, 1.8, -1.6)$. In the driven system for system (35), we define that the jump threshold $V_j = 3$. The initial conditions are $D_R^{q-1}v_i(s) = (-0.6, 0.6, 1.2), s \in [-\tau_{max}, 0]$. Figure 2 shows the neurons' trajectory plots of the driven and response FMNNs, as well as the error FMNNs, without the effect of the controller. The neurons' trajectories of the driven–response FMNNs do not overlap in Figure 2a, and the ones in the error FMNNs exhibit sustained oscillations in Figure 2b, which indicate the absence of asymptotic synchronization between the driven–response systems without control intervention. In the following, the choice of appropriate controller parameters will enable the systems to achieve asymptotic synchronization.



Figure 2. Trajectory plots of FMNNs without the effect of the proposed controller. (a) Trajectory plots of driven FMNNs ($\mu(t)$) and response FMNNs ($\nu(t)$); (b) trajectory plots of error FMNNs ($\zeta(t)$.

The proposed controller is like (9) with the static event trigger strategy, and the parameters are discussed as follows. Through simple calculation, we could obtain $\Xi_1 - \mathfrak{p}_{11} = -4.5488$, $\Xi_2 - \mathfrak{p}_{12} = -2.9958$, and $\Xi_3 - \mathfrak{p}_{13} = -3.2408$ and $\Theta_1 - \bar{\mathfrak{p}}_{21} = -3.6640$, $\Theta_2 - \bar{\mathfrak{p}}_{22} = -3.1780$, and $\Theta_3 - \bar{\mathfrak{p}}_{23} = -2.8040$. The feedback gains of controllers $\psi_i(t)$ are $\mathfrak{p}_{11} = 4.7$, $\mathfrak{p}_{12} = 3.1$, $\mathfrak{p}_{13} = 3.5$, and

$$\begin{cases} p_{21} = 4.5, & (\text{sgn}(\zeta_1(t_{k-1}))) / (\text{sgn}(\zeta_1(t))) > 0, \\ p_{21} = -4.5, & others, \\ p_{22} = 4, & (\text{sgn}(\zeta_2(t_{k-1}))) / (\text{sgn}(\zeta_2(t))) > 0, \\ p_{22} = -4, & others, \\ p_{23} = 3.5, & (\text{sgn}(\zeta_3(t_{k-1}))) / (\text{sgn}(\zeta_3(t))) > 0, \\ p_{23} = -3.5, & others. \end{cases}$$

And we chose $\iota = 0.8$ and $\varsigma = 0.6$. The impulsive strengths are

$$\begin{aligned} \alpha_{1k}(x(t)) &= [0.7, 0.1, 0.1] \cdot [x_1(t), \sin(x_2(t)), \tanh(x_3(t))], \\ \alpha_{2k}(x(t)) &= [0.7, 0.1, 0.1] \cdot [x_2(t), \sin(x_3(t)), \tanh(x_1(t))], \\ \alpha_{3k}(x(t)) &= [0.7, 0.1, 0.1] \cdot [x_3(t), \sin(x_1(t)), \tanh(x_2(t))], \end{aligned}$$

which satisfy Theorem 1. Thus, one obtains that $\sum_{j=1}^{n} \alpha_{1j}^{k} = \sum_{j=1}^{n} \alpha_{2j}^{k} = \sum_{j=1}^{n} \alpha_{3j}^{k} = 0.7$. As per Theorem 1, the driven system (35) and its response system will asymptotically synchronize under the effect of controller $\psi_i(t)$. Figure 3 illustrates the neurons' trajectory plots of the driven-response FMNNs, as well as the error FMNNs, with the action of controller (9) utilizing the static trigger strategy. From Figure 3a, it can be observed that after $t \approx 0.7$, the trajectories of the

response FMNNs and the driven FMNNs coincide, indicating that they have achieved asymptotic synchronization. Moreover, in Figure 3b, the error FMNNs also stabilize at the same moment, further demonstrating the effectiveness of the controller. Figure 4 illustrates the time intervals between adjacent trigger instants of the designed controller, denoted as $t_k - t_{k-1}$. It can be observed that the time intervals between adjacent trigger instants are always greater than 0, demonstrating that this controller can avoid Zeno behavior based on Definition 2.



Figure 3. Trajectory plots of FMNNs under the effect of controller (9) with static trigger condition (13) in Example 1. (a) Trajectory plots of driven FMNNs ($\mu(t)$) and response FMNNs ($\nu(t)$); (b) trajectory plots of error FMNNs ($\zeta(t)$).



Figure 4. Time intervals for controller (9) under static condition (13) in Example 1.

Example 2. In terms of the dynamic event trigger strategy, the driven and response FMNNs with mixed delays and jump mismatches are the same as in Example 1. And the parameters used for controller (9) are p_{1i} and p_{2i} in Example 1. The initial value of the auxiliary dynamic function is $\omega(0) = 0.08$. As shown in Figure 5, the driven–response system achieves asymptotic synchronization at $t \approx 1.4$. And Zeno behavior could be removed, which could be proven in Figure 6. And the value of the auxiliary dynamic function is always greater than 0 in Figure 7, which coincides with Theorem 3. In addition, compared to the time intervals in Figure 4, it could be observed that the controller's trigger frequency in Figure 6 can be greatly decreased by inserting an auxiliary dynamic function, but at the cost of a longer period to achieve progressive synchronization owing to



the more "relaxed" trigger condition. However, this conclusion is based solely on empirical evidence from simulation experiments. The theoretical rigor of this phenomenon remains to be established, which is a question we need to address in our future work.

Figure 5. Trajectory plots of FMNNs under the effect of controller (9) with dynamic trigger condition (30) in Example 2. (a) Trajectory plots of driven FMNNs ($\mu(t)$) and response FMNNs ($\nu(t)$); (b) trajectory plots of error FMNNs ($\zeta(t)$).



Figure 6. Time intervals for controller (9) under dynamic trigger condition (30) in Example 2.



Figure 7. The value of auxiliary dynamic function (28) in Example 2.

6. Conclusions

The present study introduced a novel mathematical model for FMNNs that incorporates mixed time delays and jump mismatches. Subsequently, we designed two innovative hybrid impulsive controllers based on static or dynamic event-triggering mechanisms. In leveraging inequality techniques and impulsive analysis, synchronization criteria for the investigated systems weare derived by constructing novel Lyapunov functions. Theoretically, the design of these controllers effectively overcomes the substantial control cost challenges identified in prior research, providing crucial insights for real-world applications.

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Abbreviations

The following abbreviations are used in this manuscript:

MNNs memristive neural networks;

FMNNs fractional memristive neural networks.

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