fractal and fractional

MDPI

Article

Further Fractional Hadamard Integral Inequalities Utilizing Extended Convex Functions

Areej A. Almoneef [1,*], Mohamed A. Barakat [2,*] and Abd-Allah Hyder [3]

1 Department of Mathematical Sciences, College of Science, Princess Nourah Bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia
2 Department of Computer Science, College of Al Wajh, University of Tabuk, P.O. Box 741, Tabuk 71491, Saudi Arabia
3 Department of Mathematics, College of Science, King Khalid University, P.O. Box 9004, Abha 61413, Saudi Arabia; abahahmed@kku.edu.sa
* Correspondence: aaalmoneef@pnu.edu.sa (A.A.A.); m.barakat@ut.edu.sa (M.A.B.)

Abstract: This work investigates novel fractional Hadamard integral inequalities by utilizing extended convex functions and generalized Riemann-Liouville operators. By carefully using extended integral formulations, we not only find novel inequalities but also improve the accuracy of error bounds related to fractional Hadamard integrals. Our study broadens the applicability of these inequalities and shows that they are useful for a variety of convexity cases. Our results contribute to the advancement of mathematical analysis and provide useful information for theoretical comprehension as well as practical applications across several scientific directions.

Keywords: Hadamard inequalities; fractional integral operators; convex functions; error bounds

MSC: 26D15; 26D10; 26D07

check for updates

Citation: Almoneef, A.A.; Barakat, M.A.; Hyder, A.-A. Further Fractional Hadamard Integral Inequalities Utilizing Extended Convex Functions. *Fractal Fract.* **2024**, *8*, 230. https://doi.org/10.3390/fractalfract8040230

Academic Editors: Ivanka Stamova, Carla M. A. Pinto and Dana Copot

Received: 2 March 2024
Revised: 8 April 2024
Accepted: 10 April 2024
Published: 16 April 2024

1. Introduction

Fractional calculus is a fascinating and burgeoning branch of mathematics that extends the concept of differentiation and integration to non-integer orders. Unlike classical calculus, which deals exclusively with integer-order differentials and integrals, fractional calculus introduces the notion of fractional derivatives and integrals, enabling a more nuanced and versatile approach to understanding complex phenomena in various fields of science and engineering [1,2]. Over the years, fractional calculus has gained prominence in diverse applications, ranging from physics and engineering to biology, finance, and control theory. Researchers have recognized its utility in modeling and analyzing complex systems characterized by memory, non-locality, and anomalous behavior. For instance, fractional calculus has been instrumental in describing the behavior of viscoelastic materials, the diffusion of contaminants in porous media, and the dynamics of financial markets [3–5].

A specialised area of mathematical analysis known as fractional integral inequalities provides a distinctive viewpoint on how convex functions and fractional calculus interact. These inequalities are crucial to many disciplines, including physics, control theory, and fractional differential equations. Caputo fractional derivative, which extends the concept of differentiation to non-integer orders, has been a focal point in the study of fractional integral inequalities involving convex functions. Research by Baleanu et al. [5] established conditions under which convexity is preserved under Caputo fractional differentiation, providing a fundamental link between fractional calculus and convex analysis. Scholars like Agarwal and Ahmad [6] have investigated fractional integral inequalities connected with convex functions. These inequalities are instrumental in establishing bounds and estimates for solutions of fractional differential equations, with implications in the analysis of complex systems. Fractional Hardy-type inequalities involving convex functions have been explored in the context of fractional Sobolev spaces. These inequalities, as demonstrated by Wang and Lou [7], provide essential tools for understanding the behavior of functions in these spaces and have applications in the study of fractional partial differential equations.

For the purpose of creating a variety of integral inequalities that primarily rely on the Hermite-Hadamard inequality, Sarikaya et al. [8] used concepts from fractional calculus. This method has given research a new direction. After that, many researchers have extensively applied the concepts of fractional calculus and obtained various fresh and inventive modifications of inequalities via convex functions and their generalisations, as seen in [9–12].

Our principal objective is to combining the generalized Riemann-Liouville operators with the extended convex functions, aiming to establish novel fractional Hadamard integral inequalities. In addition, we endeavor to furnish enhanced understanding through the provision of error bounds for a diverse range of fractional Hadamard integral inequalities, facilitated by the application of specific integral fractional formulas. Through meticulous analysis, our findings reveal numerous distinct scenarios that contains various forms of convexity. These derived insights illuminate the intricate interplay between fractional calculus and convex functions, offering valuable tools for mathematical analysis and applications across a multitude of disciplines.

This paper is organized as follows: Section 2 offers key terms and findings necessary for understanding the study's main outcomes. Section 3 contains new fractional Hadamard integral inequalities that will be investigated by combining the generalized Riemann-Liouville operators with convex functions. Section 4 gives the error bounds for the investigated fractional Hadamard integral inequalities. Section 5 includes some conclusions.

2. Pertinent Terminology and Discoveries

Convex functions are fundamental in optimization, economics, and machine learning, offering a unique global minimum and smooth behavior [12–15]. Recently, many authors formulated Hermite–Hadamard inequalities by employing the theory of convex functions in conjunction with Riemann–Liouville fractional integrals [16–19].

Before moving on to the study's main findings, it would be appropriate to explain some of the pertinent terminology and discoveries.

Definition 1 ([20]). *A function $Z : [g,h] \to \mathbb{R}$ is regarded to be convex if it meets the afterwards inequality.*

$$Z(\rho\sigma_1 + (1-\rho)\sigma_2) \leq \rho Z(\sigma_1) + (1-\rho)Z(\sigma_2), \;\; \rho \in [0,1], \text{ and } \sigma_1, \sigma_2 \in [g,h]. \tag{1}$$

If Z's additive inverse is convex, then Z is deemed to have a concave form.

The following Hermite-Hadamard inequality shows a tangible geometric illustration of a convex function.

Theorem 1 ([21]). *We obtain the succeeding inequality if $Z : [g,h] \to \mathbb{R}$ is a convex function.*

$$Z\left(\frac{g+h}{2}\right) \leq \frac{1}{h-g}\int_g^h Z(\mu)d\mu \leq \frac{Z(g)+Z(h)}{2}. \tag{2}$$

if Z is concave, the inequality (2) can be satisfied and oriented in the opposite direction.

Many researchers have looked into the characteristics and applications of the following types of convex functions.

Definition 2 ([20]). *A function $Z : [g,h] \to \mathbb{R}$ is called χ-convex if it meets the afterwards inequality.*

$$Z(\rho\sigma_1 + \chi(1-\rho)\sigma_2) \leq \rho Z(\sigma_1) + \chi(1-\rho)Z(\sigma_2), \;\; \rho, \chi \in [0,1], \text{ and } \sigma_1, \sigma_2 \in [g,h]. \tag{3}$$

Definition 3 ([20])**.** *A function $Z : [g,h] \to \mathbb{R}$ is said to be (v,χ)-convex if it meets the following inequality.*

$$Z(\rho\sigma_1 + \chi(1-\rho)\sigma_2) \leq \rho^v Z(\sigma_1) + \chi(1-\rho^v)Z(\sigma_2), \ \ \rho,\chi,v \in [0,1], \text{ and } \sigma_1,\sigma_2 \in [g,h]. \quad (4)$$

Below represent a few core concepts and rules for fractional calculus which will be utilised in the present study.

Definition 4 ([22])**.** *Let $Z \in L^1[g,h]$ with $g < h$. The Riemann-Liouville $\varkappa$-order integrals $\Lambda^{\varkappa}_{g+}Z$ and $\Lambda^{\varkappa}_{h-}Z$ are postulated as*

$$\Lambda^{\varkappa}_{g+}\, Z(\sigma) = \frac{1}{\Gamma(\varkappa)}\int_g^{\sigma} (\sigma-\mu)^{\varkappa-1} Z(\mu)\, d\mu, \quad \sigma > g, \tag{5}$$

$$\Lambda^{\varkappa}_{h-}\, Z(\sigma) = \frac{1}{\Gamma(\varkappa)}\int_{\sigma}^{h} (\mu-\sigma)^{\varkappa-1} Z(\mu)\, d\mu, \quad \sigma < h, \tag{6}$$

respectively. Here, $\Gamma(\varkappa) = \int_0^{\infty} \mu^{\varkappa-1}\exp(\mu) d\mu$, and $\Lambda^0_{g+}\, Z(\sigma) = \Lambda^0_{h-}\, Z(\sigma) = Z(\sigma)$.

The comprehensive fractional integrals below were offered by Jarad et al. [23]. Additionally, they established some characteristics and correlations to a variety of fractional integrals.

Definition 5 ([23])**.** *Let $\varkappa > 0$ and $\delta \in (0,1]$. For $Z \in L^1[g,h]$, the generalized fractional Riemann-Liouville integrals ${}^{\varkappa}\Lambda^{\delta}_{g}Z$ and ${}^{\varkappa}\Lambda^{\delta}_{h-}Z$ are defined by*

$${}^{\varkappa}\Lambda^{\delta}_{g+}Z(\sigma) = \frac{1}{\Gamma(\varkappa)}\int_g^{\sigma}\left(\frac{(\sigma-\mu)^{\delta}-(\mu-g)^{\delta}}{\delta}\right)^{\varkappa-1}\frac{Z(\mu)}{(\mu-g)^{1-\delta}}\, d\mu, \quad \sigma > g, \tag{7}$$

and

$${}^{\varkappa}\Lambda^{\delta}_{h-}Z(\sigma) = \frac{1}{\Gamma(\varkappa)}\int_{\sigma}^{h}\left(\frac{(h-\sigma)^{\delta}-(h-\mu)^{\delta}}{\delta}\right)^{\varkappa-1}\frac{Z(\mu)}{(h-\mu)^{1-\delta}}\, d\mu, \quad \sigma < h, \tag{8}$$

respectively.

Sarikaya et al. [8] start with the next fascinating Hermite-Hadamard inequality, which is addressed using Riemann-Liouville integrals (5) and (6).

Theorem 2. *Assume $Z : [g,h] \to \mathbb{R}$ is convex with $Z \in L_1[g,h]$ and $Z > 0$. Then for $\varkappa > 0$, we have*

$$\begin{aligned} Z\left(\frac{g+h}{2}\right) &\leq \frac{\Gamma(\varkappa+1)}{2(h-g)^{\delta}}\left[\Lambda^{\varkappa}_{g+}Z(h) + \Lambda^{\varkappa}_{h-}Z(g)\right] \\ &\leq \frac{Z(g)+Z(h)}{2}. \end{aligned} \tag{9}$$

Furthermore, Sarikaya and Yldrm [18] present the afterwards Hermite-Hadamard sort inequality for the operators (5) and (6).

Theorem 3. *Assume $Z : [g,h] \to \mathbb{R}$ is convex with $Z \in L_1[g,h]$, $Z > 0$ and $\delta \in (0,1]$. Then the next fractional inequalities are correct:*

$$\begin{aligned} Z\left(\frac{g+h}{2}\right) &\leq \frac{2^{\delta-1}\Gamma(\delta+1)}{(h-g)^{\delta}}\left[\Lambda^{\delta}_{\left(\frac{g+h}{2}\right)^+}Z(h) + \Lambda^{\delta}_{\left(\frac{g+h}{2}\right)^-}Z(g)\right] \\ &\leq \frac{Z(h)+Z(g)}{2}. \end{aligned}$$

Set et al. [19] put forward a noteworthy Hermite–Hadamard inequality utilizing the fractional integrals in (7) and (8). This inequality is as follows:

Theorem 4. *Consider Z to be a convex positive function from $[g,h]$ to $\mathbb{R}$ with $Z \in L_1[g,h]$ and $0 \le \rho \le \sigma$. Then the fractional integrals ${}^{\varkappa}\Lambda^{\delta}_{g+}$ and ${}^{\varkappa}\Lambda^{\delta}_{h-}$ satisfy the following inequality.*

$$Z\left(\frac{g+h}{2}\right) \le \frac{\Gamma(\varkappa+1)\delta^{\varkappa}}{2(h-g)^{\delta\varkappa}}\left[{}^{\varkappa}\Lambda^{\delta}_{g+}Z(h) + {}^{\varkappa}\Lambda^{\delta}_{h-}Z(g)\right] \le \frac{Z(g)+Z(h)}{2}, \tag{10}$$

where $Re(\varkappa) > 0$ and $\delta \in [0,1]$.

The Hermite-Hadamard inequality of a convex and positive function containing the fractional operators (7) and (8) was also expressed by Gözpınar [24] as follows:

Theorem 5. *Assume Z is a convex and positive function from $[g,h]$ to $\mathbb{R}$ with $Z \in L_1[g,h]$. Then the fractional integrals ${}^{\varkappa}\Lambda^{\delta}_{g+}$ and ${}^{\varkappa}\Lambda^{\delta}_{h-}$ fulfill the next inequality.*

$$Z\left(\frac{g+h}{2}\right) \le \frac{2^{\varkappa\delta-1}\Gamma(\varkappa+1)\delta^{\varkappa}}{(h-g)^{\varkappa\delta}}\left[{}^{\varkappa}\Lambda^{\delta}_{\left(\frac{g+h}{2}\right)+}Z(h) + {}^{\varkappa}\Lambda^{\delta}_{\left(\frac{g+h}{2}\right)-}Z(g)\right] \le \frac{Z(g)+Z(h)}{2}. \tag{11}$$

3. Main Outcomes

This section contains main results of our study which are implemented by using the generalized Riemann-Liouville operators and the (v,χ)-convex functions.

By using the (v,χ)-convex functions and generalised types of Riemann-Liouville operators, the following result provides a novel interpretation of fractional Hadamard integral inequalities.

Theorem 6. *Let Z be a positive mapping from $[g,h]$ into $\mathbb{R}$ with $0 \le g < h$ and $Z \in L_1[g,h]$. If Z is (v,χ)-convex mapping on $[g,\chi h]$ with $v,\chi \in (0,1]$, then the following integral inequality is valid for the fractional operators ${}^{\varkappa}\Lambda^{\delta}_{g+}$ and ${}^{\varkappa}\Lambda^{\delta}_{h-}$.*

$$\begin{aligned} Z\left(\frac{g+\chi h}{2}\right) &\le \frac{\delta^{\varkappa}\Gamma(\varkappa+1)}{2^{v}(\chi h-g)^{\varkappa\delta}}\left[{}^{\varkappa}\Lambda^{\delta}_{g+}Z(\chi h) + \chi^{\varkappa\delta+1}(2^{v}-1)^{\varkappa}\Lambda^{\delta}_{h-}Z\left(\frac{g}{\chi}\right)\right] \\ &\le \frac{\varkappa\delta^{\varkappa}\left[Z(g)-\chi^{2}(2^{v}-1)Z\left(\frac{g}{\chi^{2}}\right)\right]J(v,\delta,\varkappa)}{2^{v}} \\ &+ \frac{\chi\left[Z(h)(1+\varkappa\delta^{\varkappa}(2^{v}-2)J(v,\delta,\varkappa)) + \chi(2^{v}-2)Z\left(\frac{g}{\chi^{2}}\right)\right]}{2^{v}}, \end{aligned} \tag{12}$$

where,

$$J(v,\delta,\varkappa) = \int_0^1 \mu^{v}\left(\frac{1-(1-\mu)^{\delta}}{\delta}\right)^{\varkappa-1}(1-\mu)^{\delta-1}d\mu. \tag{13}$$

Proof. From the (v,χ)-convexity of Z, we have:

$$\begin{aligned} Z\left(\frac{\phi+\chi\psi}{2}\right) &= Z\left(\frac{1}{2}\phi+\chi(1-\frac{1}{2})\psi\right) \\ &\le \frac{1}{2^{v}}Z(\phi)+\chi(1-\frac{1}{2^{v}})Z(\psi) \\ &\le \frac{Z(\phi)+\chi(2^{v}-1)Z(\psi)}{2^{v}}, \quad \phi,\psi \in [g,h]. \end{aligned} \tag{14}$$

If $\phi = g\mu + \chi(1-\mu)h$ and $\psi = h\mu + (1-\mu)\frac{g}{\chi}$, then inequality (14) takes the form:

$$2^{v} Z\left(\frac{g+\chi h}{2}\right) \leq Z(g\mu + \chi(1-\mu)h) + \chi(2^{v}-1) Z\left(h\mu + (1-\mu)\frac{g}{\chi}\right). \tag{15}$$

After multiplying the two sides of (15) with $\left(\frac{1-(1-\mu)^{\delta}}{\delta}\right)^{\varkappa-1}(1-\mu)^{\delta-1}$ and integrating across the interval [0,1], we obtain:

$$\begin{aligned} & 2^{v} Z\left(\frac{g+\chi h}{2}\right) \int_0^1 \left(\frac{1-(1-\mu)^{\delta}}{\delta}\right)^{\varkappa-1}(1-\mu)^{\delta-1} d\mu \\ \leq & \int_0^1 Z(g\mu + \chi(1-\mu)h) \left(\frac{1-(1-\mu)^{\delta}}{\delta}\right)^{\varkappa-1}(1-\mu)^{\delta-1} d\mu \\ + & \chi(2^{v}-1) \int_0^1 Z\left(h\mu + (1-\mu)\frac{g}{\chi}\right) \left(\frac{1-(1-\mu)^{\delta}}{\delta}\right)^{\varkappa-1}(1-\mu)^{\delta-1} d\mu. \end{aligned} \tag{16}$$

By transforming the integration variables by parts, we get:

$$\begin{aligned} \frac{2^{v}}{\varkappa \delta^{\varkappa}} Z\left(\frac{g++\chi h}{2}\right) \leq & \frac{1}{(\chi h - g)^{\varkappa\delta}} \left[\int_g^{\chi h} \left[\frac{(\chi h - g)^{\delta} - (\phi - g)^{\delta}}{\delta}\right]^{\varkappa-1} \frac{Z(\phi)}{(\phi - g)^{1-\delta}} d\phi \right. \\ + & \left. \chi^{\varkappa\delta+1}(2^{v}-1) \int_{g/\chi}^{h} \left[\frac{(h-\frac{g}{\chi})^{\delta} - (h-\psi)^{\delta}}{\delta}\right]^{\varkappa-1} \frac{Z(\psi)}{(h-\psi)^{1-\delta}} d\psi \right]. \end{aligned} \tag{17}$$

Based on Equations (7) and (8), inequality (17) is written as:

$$Z\left(\frac{g+\chi h}{2}\right) \leq \frac{\delta^{\varkappa}\Gamma(\varkappa+1)}{2^{v}(\chi h - g)^{\varkappa\delta}} \left[{}^{\varkappa}\Lambda^{\delta}_{g+} Z(\chi h) + \chi^{\varkappa\delta+1}(2^{v}-1)^{\varkappa}\Lambda^{\delta}_{h-} Z\left(\frac{g}{\chi}\right)\right]. \tag{18}$$

Once again, the next inequality holds for any (v,χ)-convex function.

$$\begin{aligned} & Z(g\mu + \chi(1-\mu)h) + \chi(2^{v}-1) Z\left(h\mu + (1-\mu)\frac{g}{\chi}\right) \\ \leq & \left[Z(g) - \chi^{2}(2^{v}-1) Z\left(\frac{g}{\chi^{2}}\right)\right]\mu^{v} + \chi[1 + (2^{v}-2)\mu^{v}] Z(h) + \chi^{2}(2^{v}-1) Z\left(\frac{g}{\chi^{2}}\right). \end{aligned} \tag{19}$$

Multiplying each side of (19) by $\left(\frac{1-(1-\mu)^{\delta}}{\delta}\right)^{\varkappa-1}(1-\mu)^{\delta-1}$ and integrating throughout from $\mu = 0$ to $\mu = 1$ yields:

$$\begin{aligned} & \int_0^1 Z(g\mu + \chi(1-\mu)h) \left(\frac{1-(1-\mu)^{\delta}}{\delta}\right)^{\varkappa-1}(1-\mu)^{\delta-1} d\mu \\ + & \chi(2^{v}-1) \int_0^1 Z\left(h\mu + (1-\mu)\frac{g}{\chi}\right) \left(\frac{1-(1-\mu)^{\delta}}{\delta}\right)^{\varkappa-1}(1-\mu)^{\delta-1} d\mu \\ \leq & \left[Z(g) - \chi^{2}(2^{v}-1) Z\left(\frac{g}{\chi^{2}}\right)\right] J(v,\delta,\varkappa) + \chi Z(h) \int_0^1 \left(\frac{1-(1-\mu)^{\delta}}{\delta}\right)^{\varkappa-1}(1-\mu)^{\delta-1} d\mu \\ + & \chi Z(h)(2^{v}-2) J(v,\delta,\varkappa) + \chi^{2}(2^{v}-1) Z\left(\frac{g}{\chi^{2}}\right) \int_0^1 \left(\frac{1-(1-\mu)^{\delta}}{\delta}\right)^{\varkappa-1}(1-\mu)^{\delta-1} d\mu, \end{aligned} \tag{20}$$

where, $J(v,\delta,\varkappa)$ is defined by (13). Through a change in variables, we get

$$\begin{aligned}
&\frac{1}{(\chi h-g)^{\varkappa\delta}}\left[\int_{g}^{\chi h}\left[\frac{(\chi h-g)^{\delta}-(\phi-g)^{\delta}}{\delta}\right]^{\varkappa-1}\frac{Z(\phi)}{(\phi-g)^{1-\delta}}d\phi\right. \\
&+\left.\chi^{\varkappa\delta+1}(2^{v}-1)\int_{g/\chi}^{h}\left[\frac{(h-\frac{g}{\chi})^{\delta}-(h-\psi)^{\delta}}{\delta}\right]^{\varkappa-1}\frac{Z(\psi)}{(h-\psi)^{1-\delta}}d\psi\right] \\
&\le \left[Z(g)-\chi^{2}(2^{v}-1)Z\left(\frac{g}{\chi^{2}}\right)\right]J(v,\delta,\varkappa)+\frac{\chi}{\varkappa\delta^{\varkappa}}Z(h) \\
&+ \chi Z(h)(2^{v}-2)J(v,\delta,\varkappa)+\frac{\chi^{2}(2^{v}-1)}{\varkappa\delta^{\varkappa}}Z\left(\frac{g}{\chi^{2}}\right).
\end{aligned} \tag{21}$$

Inequality (21) may be expressed as follows when Equations (7) and (8) are used.

$$\begin{aligned}
&\frac{\delta^{\varkappa}\Gamma(\varkappa+1)}{2^{v}(\chi h-g)^{\varkappa\delta}}\left[{}^{\varkappa}\Lambda^{\delta}_{g+}Z(\chi h)+\chi^{\varkappa\delta+1}(2^{v}-1)^{\varkappa}\Lambda^{\delta}_{h-}Z\left(\frac{g}{\chi}\right)\right] \\
&\le \frac{\varkappa\delta^{\varkappa}\left[Z(g)-\chi^{2}(2^{v}-1)Z\left(\frac{g}{\chi^{2}}\right)\right]J(v,\delta,\varkappa)}{2^{v}} \\
&+ \frac{\chi[Z(h)(1+\varkappa\delta^{\varkappa}(2^{v}-2)J(v,\delta,\varkappa))]+\chi(2^{v}-1)Z\left(\frac{g}{\chi^{2}}\right)}{2^{v}}.
\end{aligned} \tag{22}$$

According to (18) and (22), inequality (12) can be obtained. □

Corollary 1. *If we set $v=1$ in (12), we are able to derive the following inequality for functions that are χ-convex.*

$$\begin{aligned}
Z\left(\frac{g+\chi h}{2}\right) &\le \frac{\delta^{\varkappa}\Gamma(\varkappa+1)}{2(\chi h-g)^{\varkappa\delta}}\left[{}^{\varkappa}\Lambda^{\delta}_{g+}Z(\chi h)+\chi^{\varkappa\delta+1\varkappa}\Lambda^{\delta}_{h-}Z\left(\frac{g}{\chi}\right)\right] \\
&\le \frac{\varkappa\delta^{\varkappa}\left[Z(g)-\chi^{2}Z\left(\frac{g}{\chi^{2}}\right)\right]J(1,\delta,\varkappa)}{2}+\frac{\chi}{2}\left[Z(h)+\chi Z\left(\frac{g}{\chi^{2}}\right)\right].
\end{aligned} \tag{23}$$

Corollary 2. *If we select $\chi=1$, we get the subsequent inequality for v-convex functions.*

$$\begin{aligned}
Z\left(\frac{g+h}{2}\right) &\le \frac{\delta^{\varkappa}\Gamma(\varkappa+1)}{2^{v}(h-g)^{\varkappa\delta}}\left[{}^{\varkappa}\Lambda^{\delta}_{g+}Z(h)+(2^{v}-1)^{\varkappa}\Lambda^{\delta}_{h-}Z(g)\right] \\
&\le \frac{\varkappa\delta^{\varkappa}Z(g)(2-2^{v})J(v,\delta,\varkappa)}{2^{v}} \\
&+ \frac{Z(h)(1+\varkappa\delta^{\varkappa}(2^{v}-2)J(v,\delta,\varkappa))+(2^{v}-2)Z(g)}{2^{v}}.
\end{aligned} \tag{24}$$

Remark 1. *The following is what we get if we choose certain specific values for the factors.*

- *If we set $\delta=v=1$ in (12), then we get the result [25] (Theorem 2.1) for χ-convex functions.*
- *If we set $\delta=v=\chi=1$ in (12), then we have the outcome [8] (Theorem 2) for convex functions.*
- *If $\delta=v=\chi=\varkappa=1$, then we get inequality (2).*

Remark 2. *The integral (13) can be computed numerically for some values of the exiting parameters. See Table 1 below.*

Table 1. Some values of the integral (13) computed by *Mathematica* at some selected values of v, δ, and $\varkappa$.

v	δ	$\varkappa$	$J(v,\delta,\varkappa)$	v	δ	$\varkappa$	$J(v,\delta,\varkappa)$	v	δ	$\varkappa$	$J(v,\delta,\varkappa)$
0.2	0.2	0.5	3.81577	0.2	0.2	0.2	4.30767	0.2	0.2	0.7	4.00688
0.2	0.5	0.5	2.20942	0.2	0.5	0.2	3.19027	0.2	0.5	0.7	1.95826
0.2	0.7	0.5	1.79228	0.2	0.7	0.2	2.83905	0.2	0.7	0.7	1.49141
0.5	0.2	0.5	3.28397	0.5	0.2	0.2	3.13166	0.5	0.2	0.7	3.62537
0.5	0.5	0.5	1.72386	0.5	0.5	0.2	2.05838	0.5	0.5	0.7	1.62517
0.5	0.7	0.5	1.33194	0.5	0.7	0.2	1.73306	0.5	0.7	0.7	1.18335
0.7	0.2	0.5	3.06008	0.7	0.2	0.2	2.75863	0.7	0.2	0.7	3.44755
0.7	0.5	0.5	1.52603	0.7	05	0.2	1.70917	0.7	0.5	0.7	1.47524
0.7	0.7	0.5	1.14804	0.7	0.7	0.2	1.39758	0.7	0.7	0.7	1.04756

The next result is an alternative formulation of the Hadamard inequality for extended Riemann-Liouville fractional integrals of strongly (v,χ)-convex functions.

Theorem 7. *Suppose Z be a positive mapping from $[g,h]$ into $\mathbb{R}$, $0 \le g < h$, and $Z \in L_1[g,h]$. If Z is (v,χ)-convex mapping on $[g,\chi h]$ with $v,\chi \in (0,1]$, then the following integral inequality is satisfied for the fractional operators (7) and (8).*

$$\begin{aligned} Z\left(\frac{g+\chi h}{2}\right) &\le \frac{2^{\varkappa\delta - v}\delta^{\varkappa}\Gamma(\varkappa+1)}{(\chi h - g)^{\varkappa\delta}}\left[{}^{\varkappa}\Lambda^{\delta}_{\left(\frac{g+\chi h}{2}\right)+} Z(\chi h) + \chi^{\varkappa\delta+1}(2^v-1)^{\varkappa}\Lambda^{\delta}_{\left(\frac{g+\chi h}{2\chi}\right)-} Z\left(\frac{g}{\chi}\right)\right] \\ &\le \frac{\varkappa\delta^{\varkappa}\left[Z(g) - \chi^2(2^v-1)Z\left(\frac{g}{\chi^2}\right)\right]J(v,\delta,\varkappa)}{2^{2v}} \\ &+ \frac{\chi\left[Z(h)\left(1+\varkappa\delta^{\varkappa}(1-2^{1-v})J(v,\delta,\varkappa)\right)\right] + \chi^2(2^v-1)Z\left(\frac{g}{\chi^2}\right)}{2^v}. \end{aligned} \tag{25}$$

where, $J(v,\delta,\varkappa)$ is defined by (13).

Proof. According to the (v,χ) convexity of Z, and assuming $\phi = g\frac{\mu}{2} + \chi(1-\frac{\mu}{2})h$, $\psi = (1-\frac{\mu}{2})\frac{g}{\chi} + h\frac{\mu}{2}$ in (14), we get

$$2^v Z\left(\frac{g+\chi h}{2}\right) \le Z\left(g\frac{\mu}{2} + \chi\left(1-\frac{\mu}{2}\right)h\right) + \chi(2^v-1)Z\left(\left(1-\frac{\mu}{2}\right)\frac{g}{\chi} + h\frac{\mu}{2}\right). \tag{26}$$

After performing the operations of multiplying both sides of (26) by $\left(\frac{1-(1-\mu)^{\delta}}{\delta}\right)^{\varkappa-1}(1-\mu)^{\delta-1}$ and integrating during [0,1], the following is what we get:

$$\begin{aligned} &2^v Z\left(\frac{g+\chi h}{2}\right)\int_0^1 \left(\frac{1-(1-\mu)^{\delta}}{\delta}\right)^{\varkappa-1}(1-\mu)^{\delta-1}d\mu \\ &\le \int_0^1 Z\left(g\frac{\mu}{2} + \chi\left(1-\frac{\mu}{2}\right)h\right)\left(\frac{1-(1-\mu)^{\delta}}{\delta}\right)^{\varkappa-1}(1-\mu)^{\delta-1}d\mu \\ &+ \chi(2^v-1)\int_0^1 Z\left(\left(1-\frac{\mu}{2}\right)\frac{g}{\chi} + h\frac{\mu}{2}\right)\left(\frac{1-(1-\mu)^{\delta}}{\delta}\right)^{\varkappa-1}(1-\mu)^{\delta-1}d\mu. \end{aligned} \tag{27}$$

When we integrate by parts and then change the variables, we obtain

$$\frac{2^{v}}{\varkappa\delta^{\varkappa}}Z\left(\frac{g++\chi h}{2}\right) \leq \frac{2^{\varkappa\delta}}{(\chi h-g)^{\varkappa\delta}}\left[\int_{\frac{g+\chi h}{2}}^{\chi h}\left[\frac{\left(\chi h-\frac{g+\chi h}{2}\right)^{\delta}-\left(\phi-\frac{g+\chi h}{2}\right)^{\delta}}{\delta}\right]^{\varkappa-1}\frac{Z(\phi)}{\left(\phi-\frac{g+\chi h}{2}\right)^{1-\delta}}d\phi\right.$$
$$\left.+\ \chi^{\varkappa\delta+1}(2^{v}-1)\int_{g/\chi}^{\frac{g+\chi h}{2\chi}}\left[\frac{\left(\frac{g+\chi h}{2\chi}-\frac{g}{\chi}\right)^{\delta}-\left(\frac{g+\chi h}{2\chi}-\psi\right)^{\delta}}{\delta}\right]^{\varkappa-1}\frac{Z(\psi)}{\left(\frac{g+\chi h}{2\chi}-\psi\right)^{1-\delta}}d\psi\right]. \tag{28}$$

Using Equations (7) and (8), we can rewrite inequality (28) as:

$$Z\left(\frac{g+\chi h}{2}\right) \leq \frac{2^{\varkappa\delta-v}\delta^{\varkappa}\Gamma(\varkappa+1)}{(\chi h-g)^{\varkappa\delta}}\left[{}^{\varkappa}\Lambda^{\delta}_{\left(\frac{g+\chi h}{2}\right)+}Z(\chi h)+\chi^{\varkappa\delta+1}(2^{v}-1)^{\varkappa}\Lambda^{\delta}_{\left(\frac{g+\chi h}{2\chi}\right)-}Z\left(\frac{g}{\chi}\right)\right]. \tag{29}$$

In addition, any (v,χ)-convex function fulfills the next inequality.

$$Z\left(g\frac{\mu}{2}+\chi\left(1-\frac{\mu}{2}\right)h\right)+\chi(2^{v}-1)Z\left(\left(1-\frac{\mu}{2}\right)\frac{g}{\chi}+h\frac{\mu}{2}\right) \tag{30}$$
$$\leq \left[Z(g)-\chi^{2}(2^{v}-1)Z\left(\frac{g}{\chi^{2}}\right)\right]\left(\frac{\mu}{2}\right)^{v}+\chi\left[1+(2^{v}-2)\left(\frac{\mu}{2}\right)^{v}\right]Z(h)+\chi^{2}(2^{v}-1)Z\left(\frac{g}{\chi^{2}}\right).$$

After multiplying the two sides of (30) with $\left(\frac{1-(1-\mu)^{\delta}}{\delta}\right)^{\varkappa-1}(1-\mu)^{\delta-1}$ and integrating across the interval [0,1], we obtain:

$$\int_{0}^{1}Z\left(g\frac{\mu}{2}+\chi\left(1-\frac{\mu}{2}\right)h\right)\left(\frac{1-(1-\mu)^{\delta}}{\delta}\right)^{\varkappa-1}(1-\mu)^{\delta-1}d\mu \tag{31}$$
$$+\ \chi(2^{v}-1)\int_{0}^{1}Z\left(\left(1-\frac{\mu}{2}\right)\frac{g}{\chi}+h\frac{\mu}{2}\right)\left(\frac{1-(1-\mu)^{\delta}}{\delta}\right)^{\varkappa-1}(1-\mu)^{\delta-1}d\mu$$
$$\leq\ 2^{-v}\left[Z(g)-\chi^{2}(2^{v}-1)Z\left(\frac{g}{\chi^{2}}\right)\right]J(v,\delta,\varkappa)+\chi Z(h)\int_{0}^{1}\left(\frac{1-(1-\mu)^{\delta}}{\delta}\right)^{\varkappa-1}(1-\mu)^{\delta-1}d\mu$$
$$+\ \chi Z(h)(1-2^{1-v})J(v,\delta,\varkappa)+\chi^{2}(2^{v}-1)Z\left(\frac{g}{\chi^{2}}\right)\int_{0}^{1}\left(\frac{1-(1-\mu)^{\delta}}{\delta}\right)^{\varkappa-1}(1-\mu)^{\delta-1}d\mu,$$

When integrating by parts and changing the variables, the result is:

$$\frac{2^{\varkappa\delta}}{(\chi h-g)^{\varkappa\delta}}\left[\int_{\frac{g+\chi h}{2}}^{\chi h}\left[\frac{\left(\chi h-\frac{g+\chi h}{2}\right)^{\delta}-\left(\phi-\frac{g+\chi h}{2}\right)^{\delta}}{\delta}\right]^{\varkappa-1}\frac{Z(\phi)}{\left(\phi-\frac{g+\chi h}{2}\right)^{1-\delta}}d\phi\right. \tag{32}$$
$$\left.+\ \chi^{\varkappa\delta+1}(2^{v}-1)\int_{g/\chi}^{\frac{g+\chi h}{2\chi}}\left[\frac{\left(\frac{g+\chi h}{2\chi}-\frac{g}{\chi}\right)^{\delta}-\left(\frac{g+\chi h}{2\chi}-\psi\right)^{\delta}}{\delta}\right]^{\varkappa-1}\frac{Z(\psi)}{\left(\frac{g+\chi h}{2\chi}-\psi\right)^{1-\delta}}d\psi\right]$$
$$\leq\ 2^{-v}\left[Z(g)-\chi^{2}(2^{v}-1)Z\left(\frac{g}{\chi^{2}}\right)\right]J(v,\delta,\varkappa)+\frac{\chi}{\varkappa\delta^{\varkappa}}Z(h)$$
$$+\ \chi Z(h)(1-2^{1-v})J(v,\delta,\varkappa)+\frac{\chi^{2}(2^{v}-1)}{\varkappa\delta^{\varkappa}}Z\left(\frac{g}{\chi^{2}}\right).$$

Using Equations (7) and (8), we can rewrite (32) as follows.

$$\begin{aligned}
&\frac{2^{\varkappa\delta-v}\delta^{\varkappa}\Gamma(\varkappa+1)}{(\chi h-g)^{\varkappa\delta}}\left[{}^{\varkappa}\Lambda^{\delta}_{\left(\frac{g+\chi h}{2}\right)+}Z(\chi h)+\chi^{\varkappa\delta+1}(2^{v}-1)^{\varkappa}\Lambda^{\delta}_{\left(\frac{g+\chi h}{2\chi}\right)-}Z\left(\frac{g}{\chi}\right)\right] \\
\leq\;& \frac{\varkappa\delta^{\varkappa}\left[Z(g)-\chi^{2}(2^{v}-1)Z\left(\frac{g}{\chi^{2}}\right)\right]J(v,\delta,\varkappa)}{2^{2v}} \\
+\;& \frac{\chi\left[Z(h)\left(1+\varkappa\delta^{\varkappa}(1-2^{1-v})J(v,\delta,\varkappa)\right)\right]+\chi^{2}(2^{v}-1)Z\left(\frac{g}{\chi^{2}}\right)}{2^{v}}.
\end{aligned} \tag{33}$$

From inequalities (29) and (33), we deduce the required result. □

Corollary 3. *If we choose $v = 1$ in (25), we are able to deduce the upcoming inequality χ-convex functions.*

$$\begin{aligned}
Z\left(\frac{g+\chi h}{2}\right) &\leq \frac{2^{\varkappa\delta-1}\delta^{\varkappa}\Gamma(\varkappa+1)}{(\chi h-g)^{\varkappa\delta}}\left[{}^{\varkappa}\Lambda^{\delta}_{\left(\frac{g+\chi h}{2}\right)+}Z(\chi h)+\chi^{\varkappa\delta+1}{}^{\varkappa}\Lambda^{\delta}_{\left(\frac{g+\chi h}{2\chi}\right)-}Z\left(\frac{g}{\chi}\right)\right] \\
&\leq \frac{\varkappa\delta^{\varkappa}\left[Z(g)-\chi^{2}Z\left(\frac{g}{\chi^{2}}\right)\right]J(1,\delta,\varkappa)}{4} \\
&+ \frac{\chi}{2}\left(Z(h)+\chi Z\left(\frac{g}{\chi^{2}}\right)\right).
\end{aligned} \tag{34}$$

Corollary 4. *If we put $\chi = 1$ in (25), we acquire the next inequality for v-convex functions.*

$$\begin{aligned}
Z\left(\frac{g+h}{2}\right) &\leq \frac{2^{\varkappa\delta-v}\delta^{\varkappa}\Gamma(\varkappa+1)}{(h-g)^{\varkappa\delta}}\left[{}^{\varkappa}\Lambda^{\delta}_{\left(\frac{g+h}{2}\right)+}Z(h)+(2^{v}-1)^{\varkappa}\Lambda^{\delta}_{\left(\frac{g+h}{2}\right)-}Z(g)\right] \\
&\leq Z(g)\left(\frac{\varkappa\delta^{\varkappa}(2-2^{v})J(v,\delta,\varkappa)+2^{v}(2^{v}-1)}{2^{2v}}\right) \\
&+ Z(h)\left(\frac{1+\varkappa\delta^{\varkappa}(1-2^{1-v})J(v,\delta,\varkappa)}{2^{v}}\right).
\end{aligned} \tag{35}$$

Corollary 5. *If we set $v = \chi = 1$ in (25), we acquire the next inequality for convex functions.*

$$\begin{aligned}
Z\left(\frac{g+h}{2}\right) &\leq \frac{2^{\varkappa\delta-v}\delta^{\varkappa}\Gamma(\varkappa+1)}{(h-g)^{\varkappa\delta}}\left[{}^{\varkappa}\Lambda^{\delta}_{\left(\frac{g+h}{2}\right)+}Z(h)+{}^{\varkappa}\Lambda^{\delta}_{\left(\frac{g+h}{2}\right)-}Z(g)\right] \\
&\leq \left(\frac{Z(g)+Z(h)}{2}\right).
\end{aligned} \tag{36}$$

Remark 3. *If $\delta = 1$ in (13), then $J(v,\delta,\varkappa) = \int_0^1 \mu^{\varkappa+v-1}d\mu = \frac{1}{\varkappa+v}$. Under this assumption, the upcoming connections can be observed.*

- *If we put $\delta = 1$ in (34), we get the result [25] (Theorem 2.6).*
- *If we set $\delta = 1$ in (36), we get the finding [18] (Theorem 4).*
- *If we put $\delta = \varkappa = 1$ in (36), we get Theorem 1.*

4. Error Limitations for (v, χ)-Convex Functions

In this part, we refine the precision of error bounds associated with the derived fractional Hadamard inequalities by carefully applying extended integral formulations.

Lemma 1. *Let Z be a positive differentiable mapping from $[g,h]$ into $\mathbb{R}$ with $0 \le g < h$ and $Z \in L_1[g,h]$.Consequently, the subsequent integral equality is true.*

$$\frac{Z(g)+Z(h)}{\delta^{\varkappa}} - \frac{\Gamma(\varkappa+1)}{(h-g)^{\varkappa\delta}}\left[(\Lambda^{\delta}_{g^-}Z)(h) + (\Lambda^{\delta}_{h^+}Z)(g)\right]$$
$$= (h-g)\int_0^1 \left[\left(\frac{1-(1-t)^{\delta}}{\delta}\right)^{\varkappa} - \left(\frac{1-t^{\delta}}{\delta}\right)^{\varkappa}\right] Z'(th+(1-t)g)dt. \tag{37}$$

Proof. We now start with the following integral:

$$\begin{aligned}
&\int_0^1 \left[\frac{1-t^{\delta}}{\delta}\right]^{\varkappa} Z^{'}(th+(1-t)g)dt \\
&= \frac{-Z(g)}{\delta^{\varkappa}(h-g)} - \frac{\varkappa}{(h-g)}\int_0^1 [\frac{1-t^{\delta}}{\delta}]^{\varkappa-1} Z(th+(1-t)g)\frac{dt}{t^{1-\delta}} \\
&= \frac{-Z(g)}{\delta^{\varkappa}(h-g)} + \frac{\varkappa}{(h-g)^{\varkappa\delta}}\int_g^h [\frac{(h-g)^{\delta}-(u-g)^{\delta}}{\delta}]^{\varkappa-1} Z(u)\frac{du}{(u-g)^{1-\delta}} \\
&= \frac{-Z(g)}{\delta^{\varkappa}(h-g)} + \frac{\Gamma(\varkappa+1)}{(h-g)^{\varkappa\delta+1}}\Lambda^{\varkappa}_{g^-}Z(h).
\end{aligned} \tag{38}$$

Similar to the previous equation, the following equation can be produced through integration by parts:

$$\begin{aligned}
&\int_0^1 [\frac{1-(1-t)^{\delta}}{\delta}]^{\varkappa} Z^{'}(th+(1-t)g)dt \\
&= \frac{Z(g)}{\delta^{\varkappa}(h-g)} - \frac{\Gamma(\varkappa+1)}{(h-g)^{\varkappa\delta+1}}\Lambda^{\varkappa}_{h^+}Z(h)
\end{aligned} \tag{39}$$

We can obtain the required Equation (37) by employing Equations (38) and (39). □

Lemma 2. *Let Z be a positive differentiable mapping from $[g,h]$ into $\mathbb{R}$ with $Z' \in L[g,h]$. Consequently, the subsequent integral equality is true.*

$$\begin{aligned}
&\frac{2^{\delta\varkappa-1}\Gamma(\varkappa+1)}{(\chi h-g)^{\delta\varkappa}}\left[(-1)^{\delta}(\Lambda^{\varkappa}_{\frac{g+\chi h}{2}^-}Z)(\chi h) + \chi^{\varkappa\delta+1}(\Lambda^{\varkappa}_{\frac{g+\chi h}{2\chi}^+}Z)(\frac{g}{\chi})\right] - \frac{1}{2\delta^{\varkappa}}\left[Z(\chi h) + \chi Z(\frac{g}{\chi})\right] \\
&= \frac{\chi h-g}{4}\left[-\int_0^1 \left[\frac{1-t^{\delta}}{\delta}\right]^{\varkappa} Z'(\frac{gt}{2}+\frac{\chi h(2-t)}{2})dt + \int_0^1 \left[\frac{1-t^{\delta}}{\delta}\right]^{\varkappa} Z'(\frac{g(2-t)}{2\chi}+\frac{t}{2}h)dt\right].
\end{aligned} \tag{40}$$

Proof. Let's start with

$$\begin{aligned}
&\frac{\chi h-g}{4}\int_0^1 \left[\frac{1-t^{\delta}}{\delta}\right]^{\varkappa} Z'(\frac{gt}{2}+\frac{\chi h(2-t)}{2})dt = \frac{Z(\chi h)}{2\delta^{\varkappa}} - \frac{\varkappa}{2}\int_0^1 \left[\frac{1-t^{\delta}}{\delta}\right]^{\varkappa-1}\frac{Z(\frac{gt}{2}+\frac{\chi h(2-t)}{2})}{t^{1-\delta}}dt \\
&= \frac{Z(\chi h)}{2\delta^{\varkappa}} - \frac{(-1)^{\delta}2^{\delta\varkappa-1}\varkappa}{(\chi h-g)^{\delta\varkappa}}\int_{\chi h}^{\frac{g+\chi h}{2}}\left[\frac{(\frac{g+\chi h}{2}-\chi h)^{\delta}-(u-\chi h)^{\delta}}{\delta}\right]^{\varkappa-1}\frac{Z(u)}{(u-\chi h)^{1-\delta}}du \\
&= \frac{Z(\chi h)}{2\delta^{\varkappa}} - \frac{(-1)^{\delta}2^{\delta\varkappa-1}\Gamma(\varkappa+1)}{(\chi h-g)^{\delta\varkappa}}(\Lambda^{\varkappa}_{\frac{g+\chi h}{2}^-}Z)(\chi h).
\end{aligned} \tag{41}$$

Similarly,

$$\frac{\chi h-g}{4}\int_0^1\left[\frac{1-t^\delta}{\delta}\right]^\varkappa Z'\left(\frac{g(2-t)}{2\chi}+\frac{th}{2}\right)dt=-\frac{\chi Z(\frac{g}{\chi})}{2\delta^\varkappa}+\frac{2^{\delta\varkappa-1}\chi^{\varkappa\delta+1}\Gamma(\varkappa+1)}{(\chi h-g)^{\delta\varkappa}}(\Lambda^\varkappa_{\frac{g+\chi h}{2}+}Z)(\frac{g}{\chi}). \tag{42}$$

One obtains (40), by subtracting Equation (41) from (42). □

Theorem 8. *Suppose Z be a positive mapping from $[g,h]$ into $\mathbb{R}$, $0\le g<h$, and $Z'\in L_1[g,h]$. If Z' is (v,χ)-convex mapping on $[g,\chi h]$ with $v,\chi\in(0,1]$, then the following integral inequality is satisfied for the fractional operators (7) and (8).*

$$\begin{aligned}&\left|\frac{Z(g)+Z(h)}{\delta^\varkappa}-\frac{\Gamma(\varkappa+1)}{(h-g)^{\varkappa\delta}}\left[(\Lambda^\delta_{g^-}Z)(h)+(\Lambda^\delta_{h^+}Z)(g)\right]\right|\\&\le\frac{|h-g|\,|Z'(h)-\chi Z'(\frac{g}{\chi})|}{\delta^{\varkappa+1}}\left[\delta^\nu J(\varkappa,\frac{1}{\delta},\nu)+\beta(\frac{\nu+1}{\delta},\varkappa+1)\right]+\frac{2\chi|Z'(\frac{g}{\chi})|}{\delta^{\varkappa+1}}\beta(\frac{1}{\delta},\varkappa+1)\end{aligned} \tag{43}$$

Proof. According to the (v,χ) convexity of Z', and utilizing Lemma 2 , and Equation (13), we have

$$\begin{aligned}&\left|\frac{Z(g)+Z(h)}{\delta^\varkappa}-\frac{\Gamma(\varkappa+1)}{(h-g)^{\varkappa\delta}}\left[(\Lambda^\delta_{g^-}Z)(h)+(\Lambda^\delta_{h^+}Z)(g)\right]\right|\\
&\le|h-g|\int_0^1\left|\left(\frac{1-(1-t)^\delta}{\delta}\right)^\varkappa-\left(\frac{1-t^\delta}{\delta}\right)^\varkappa\right||Z'(th+(1-t)g)|\,dt\\
&\le|h-g|\left[|Z'(h)|\int_0^1\left|\left(\frac{1-(1-t)^\delta}{\delta}\right)^\varkappa-\left(\frac{1-t^\delta}{\delta}\right)^\varkappa\right|t^\nu dt\right.\\
&\left.+\chi|Z'(\frac{g}{\chi})|\int_0^1\left|\left(\frac{1-(1-t)^\delta}{\delta}\right)^\varkappa-\left(\frac{1-t^\delta}{\delta}\right)^\varkappa\right|(1-t^\nu)dt\right]\\
&\le|h-g|\,|Z'(h)|\left[\int_0^1\left(\frac{1-(1-t)^\delta}{\delta}\right)^\varkappa t^\nu dt+\int_0^1\left(\frac{1-t^\delta}{\delta}\right)^\varkappa t^\nu dt\right]\\
&+\chi|Z'(\frac{g}{\chi})|\left[\int_0^1\left(\frac{1-(1-t)^\delta}{\delta}\right)^\varkappa(1-t^\nu)dt+\int_0^1\left(\frac{1-t^\delta}{\delta}\right)^\varkappa(1-t^\nu)dt\right]\end{aligned}$$

$$\leq \frac{|h-g|\,|Z'(h)|}{\delta^{\varkappa+1}}\left[\delta^{\nu}\int_0^1 \mu^{\varkappa}\left(\frac{1-(1-\mu)^{\frac{1}{\delta}}}{\delta}\right)^{\nu}\frac{d\mu}{(1-\mu)^{1-\frac{1}{\delta}}}+\int_0^1(1-\mu)^{\varkappa}\mu^{\frac{\nu+1}{\delta}-1}d\mu\right]$$
$$+\chi|Z'(\frac{g}{\chi})|\left[\int_0^1\left(\frac{1-(1-t)^{\delta}}{\delta}\right)^{\varkappa}(1-t^{\nu})dt+\int_0^1\left(\frac{1-t^{\delta}}{\delta}\right)^{\varkappa}(1-t^{\nu})dt\right]$$
$$\leq \frac{|h-g|\,|Z'(h)|}{\delta^{\varkappa+1}}\left[\delta^{\nu}J(\varkappa,\frac{1}{\delta},\nu)+\beta(\frac{\nu+1}{\delta},\varkappa+1)\right]+\chi|Z'(\frac{g}{\chi})|\left[\int_0^1\left(\frac{1-(1-t)^{\delta}}{\delta}\right)^{\varkappa}dt\right.$$
$$\left.-\int_0^1\left(\frac{1-(1-t)^{\delta}}{\delta}\right)^{\varkappa}t^{\nu}dt+\int_0^1\left(\frac{1-t^{\delta}}{\delta}\right)^{\varkappa}dt-\int_0^1\left(\frac{1-t^{\delta}}{\delta}\right)^{\varkappa}t^{\nu}dt\right]$$
$$\leq \frac{|h-g|\,|Z'(h)|}{\delta^{\varkappa+1}}\left[\delta^{\nu}J(\varkappa,\frac{1}{\delta},\nu)+\beta(\frac{\nu+1}{\delta},\varkappa+1)\right]$$
$$+\frac{\chi|Z'(\frac{g}{\chi})|}{\delta^{\varkappa+1}}\left[2\beta(\frac{1}{\delta},\varkappa+1)-\delta^{\nu}J(\varkappa,\frac{1}{\delta},\nu)-\beta(\frac{\nu+1}{\delta},\varkappa+1)\right]$$
$$\leq \frac{|h-g|\,(|Z'(h)|-\chi|Z'(\frac{g}{\chi})|)}{\delta^{\varkappa+1}}\left[\delta^{\nu}J(\varkappa,\frac{1}{\delta},\nu)+\beta(\frac{\nu+1}{\delta},\varkappa+1)\right]+\frac{2\chi|Z'(\frac{g}{\chi})|}{\delta^{\varkappa+1}}\beta(\frac{1}{\delta},\varkappa+1). \quad (44)$$

□

Corollary 6. *The subsequent integral inequality resulting from fractional operators (7) and (8) in sense of $\chi-$convex mapping on $[g,\chi h]$ can be derived by setting $\nu=1$ in (43).*

$$\left|\frac{Z(g)+Z(h)}{\delta^{\varkappa}}-\frac{\Gamma(\varkappa+1)}{(h-g)^{\varkappa\delta}}\left[(\Lambda^{\delta}_{g^-}Z)(h)+(\Lambda^{\delta}_{h^+}Z)(g)\right]\right|$$
$$\leq \frac{|h-g|\,|Z'(h)-\chi Z'(\frac{g}{\chi})|}{\delta^{\varkappa+1}}\left[\delta^{\nu}J(\varkappa,\frac{1}{\delta},1)+\beta(\frac{2}{\delta},\varkappa+1)\right]+\frac{2\chi|Z'(\frac{g}{\chi})|}{\delta^{\varkappa+1}}\beta(\frac{1}{\delta},\varkappa+1) \quad (45)$$

Corollary 7. *The subsequent integral inequality resulting from fractional operators (7) and (8) in sense of convex mapping on $[g,h]$ can be derived by setting $\nu=\chi=1$ in (43).*

$$\left|\frac{Z(g)+Z(h)}{\delta^{\varkappa}}-\frac{\Gamma(\varkappa+1)}{(h-g)^{\varkappa\delta}}\left[(\Lambda^{\delta}_{g^-}Z)(h)+(\Lambda^{\delta}_{h^+}Z)(g)\right]\right|$$
$$\leq \frac{|h-g|\,|Z'(h)-Z'(g)|}{\delta^{\varkappa+1}}\left[\delta^{\nu}J(\varkappa,\frac{1}{\delta},1)+\beta(\frac{2}{\delta},\varkappa+1)\right]+\frac{2|Z'(g)|}{\delta^{\varkappa+1}}\beta(\frac{1}{\delta},\varkappa+1) \quad (46)$$

Theorem 9. *Suppose Z be a positive mapping from $[g,h]$ into $\mathbb{R}$, $0\leq g<h$, and $Z'\in L_1[g,h]$. If $|Z'|^q$ is (v,χ)-convex mapping on $[g,\chi h]$ with $v,\chi\in(0,1]$, then the following integral inequality is satisfied for the fractional operators (7) and (8).*

$$\left|\frac{2^{\delta\varkappa-1}\Gamma(\varkappa+1)}{(\chi h-g)^{\delta\varkappa}}\left[(-1)^{\delta}(\Lambda^{\varkappa}_{\frac{g+\chi h}{2}^-}Z)(\chi h)+\chi^{\varkappa\delta+1}(\Lambda^{\varkappa}_{\frac{g+\chi h}{2\chi}^+}Z)(\frac{g}{\chi})\right]-\frac{1}{2\delta^{\varkappa}}\left[Z(\chi h)+\chi Z(\frac{g}{\chi})\right]\right|$$
$$\leq \frac{|\chi h-g|}{4}\left(\frac{\beta(p\varkappa+1,\frac{1}{\delta})}{\delta^{p+1}}\right)^{\frac{1}{p}}\left(\frac{1}{2^{v}(v+1)}\right)^{\frac{1}{q}}\left[|Z'(g)|+|Z'(h)|\right.$$
$$\left.+\left(\chi(2^{v}(v+1)-1)\right)^{\frac{1}{q}}\left(|Z'(h)|+|Z'(\frac{g}{\chi^2})|\right)\right]. \quad (47)$$

where β refers to beta function and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using (v,χ)-convexity of $|Z'|^q$ and Lemma 2, we obtain by applying Holder inequality

$$\left|\frac{2^{\delta\varkappa-1}\Gamma(\varkappa+1)}{(\chi h-g)^{\delta\varkappa}}\left[(-1)^{\delta}(\Lambda^{\varkappa}_{\frac{g+\chi h}{2}-}Z)(\chi h)+\chi^{\varkappa\delta+1}(\Lambda^{\varkappa}_{\frac{g+\chi h}{2\chi}+}Z)(\frac{g}{\chi})\right]-\frac{1}{2\delta^{\varkappa}}\left[Z(\chi h)+\chi Z(\frac{g}{\chi})\right]\right|$$
$$\le\frac{|\chi h-g|}{4}\left(\int_0^1\left|\frac{1-t^{\delta}}{\delta}\right|^{p\varkappa}dt\right)^{\frac{1}{p}}\left[\left(\int_0^1\left|Z'(\frac{gt}{2}+\frac{\chi h(2-t)}{2})\right|^q dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1\left|Z'(\frac{g(2-t)}{2\chi}+\frac{t}{2}h)\right|^q dt\right)^{\frac{1}{q}}\right]$$
$$\le\frac{|\chi h-g|}{4}\left(\frac{\beta(p\varkappa+1,\frac{1}{\delta})}{\delta^{p+1}}\right)^{\frac{1}{p}}\left[\left(\left|Z'(g)\right|^q\int_0^1\left(\frac{t}{2}\right)^v dt+\chi\left|Z'(h)\right|^q\int_0^1\left(1-\left(\frac{t}{2}\right)^v\right)dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\left|Z'(h)\right|^q\int_0^1\left(\frac{t}{2}\right)^v dt+\chi\left|Z'(\frac{g}{\chi^2})\right|^q\int_0^1\left(1-\left(\frac{t}{2}\right)^v\right)dt\right)^{\frac{1}{q}}\right]$$
$$\le\frac{|\chi h-g|}{4}\left(\frac{\beta(p\varkappa+1,\frac{1}{\delta})}{\delta^{p+1}}\right)^{\frac{1}{p}}\left[\left(\left|Z'(g)\right|^q\frac{1}{2^v(v+1)}+\chi\left|Z'(h)\right|^q(1-\frac{1}{2^v(v+1)})\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\left|Z'(h)\right|^q\frac{1}{2^v(v+1)}+\chi\left|Z'(\frac{g}{\chi^2})\right|^q(1-\frac{1}{2^v(v+1)})\right)^{\frac{1}{q}}\right]$$

$$\le\frac{|\chi h-g|}{4}\left(\frac{\beta(p\varkappa+1,\frac{1}{\delta})}{\delta^{p+1}}\right)^{\frac{1}{p}}\left(\frac{1}{2^v(v+1)}\right)^{\frac{1}{q}}\left[\left(\left|Z'(g)\right|^q+\chi(2^v(v+1)-1)\left|Z'(h)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\left|Z'(h)\right|^q+\chi(2^v(v+1)-1)\left|Z'(\frac{g}{\chi^2})\right|^q\right)^{\frac{1}{q}}\right]$$
$$\le\frac{|\chi h-g|}{4}\left(\frac{\beta(p\varkappa+1,\frac{1}{\delta})}{\delta^{p+1}}\right)^{\frac{1}{p}}\left(\frac{1}{2^v(v+1)}\right)^{\frac{1}{q}}\left[\left(\left(|Z'(g)|+(\chi(2^v(v+1)-1))^{\frac{1}{q}}|Z'(h)|\right)^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\left(|Z'(h)|+(\chi(2^v(v+1)-1))^{\frac{1}{q}}|Z'(\frac{g}{\chi^2})|\right)^q\right)^{\frac{1}{q}}\right]$$
$$\le\frac{|\chi h-g|}{4}\left(\frac{\beta(p\varkappa+1,\frac{1}{\delta})}{\delta^{p+1}}\right)^{\frac{1}{p}}\left(\frac{1}{2^v(v+1)}\right)^{\frac{1}{q}}\left[|Z'(g)|+|Z'(h)|\right.$$
$$\left.+\left(\chi(2^v(v+1)-1)\right)^{\frac{1}{q}}\left(|Z'(h)|+|Z'(\frac{g}{\chi^2})|\right)\right]. \tag{48}$$

□

Corollary 8. *The subsequent integral inequality resulting from fractional operators (7) and (8) in sense of $\chi-$convex mapping on $[g,\chi h]$ can be derived by setting $\nu=1$ in (47).*

$$\left|\frac{2^{\delta\varkappa-1}\Gamma(\varkappa+1)}{(\chi h-g)^{\delta\varkappa}}\left[(-1)^{\delta}(\Lambda^{\varkappa}_{\frac{g+\chi h}{2}-}Z)(\chi h)+\chi^{\varkappa\delta+1}(\Lambda^{\varkappa}_{\frac{g+\chi h}{2\chi}+}Z)(\frac{g}{\chi})\right]-\frac{1}{2\delta^{\varkappa}}\left[Z(\chi h)+\chi Z(\frac{g}{\chi})\right]\right|$$
$$\le\frac{|\chi h-g|}{4}\left(\frac{\beta(p\varkappa+1,\frac{1}{\delta})}{\delta^{p+1}}\right)^{\frac{1}{p}}\left(\frac{1}{4}\right)^{\frac{1}{q}}\left[|Z'(g)|+|Z'(h)|+3^{\frac{1}{q}}\left(|Z'(h)|+|Z'(\frac{g}{\chi^2})|\right)\right]. \tag{49}$$

Corollary 9. *The subsequent integral inequality resulting from fractional operators* (7) *and* (8) *in sense of convex mapping on* $[g, h]$ *can be derived by setting* $\nu = \chi = 1$ *in* (47)*.*

$$\left|\frac{2^{\delta\varkappa-1}\Gamma(\varkappa+1)}{h-g)^{\delta\varkappa}}\left[(-1)^{\delta}(\Lambda^{\varkappa}_{\frac{g+h}{2}^-}Z)(h)+(\Lambda^{\varkappa}_{\frac{g+h}{2}^+}Z)(g)\right]-\frac{1}{2\delta^{\varkappa}}\left[Z(h)+Z(g)\right]\right|$$
$$\leq \frac{|h-g|}{4}\left(\frac{\beta(p\varkappa+1,\frac{1}{\delta})}{\delta^{p+1}}\right)^{\frac{1}{p}}\left(\frac{1}{4}\right)^{\frac{1}{q}}\left[|Z'(g)|+|Z'(h)|+3^{\frac{1}{q}}\left(|Z'(h)|+|Z'(\frac{g}{\chi^2})|\right)\right]. \tag{50}$$

5. Conclusions

The fusion of extended convex functions, namely (υ, χ)-convex functions, with generalized Riemann-Liouville operators was done in order to reveal several new fractional Hadamard integral inequalities. Beyond this, the error bounds for the investigated fractional Hadamard integral inequalities have been presented. Additionally, there is considerable connection between the results provided and those that have already been published.

Author Contributions: Methodology and conceptualization, A.A.A., M.A.B. and A.-A.H.; data curation and writing—original draft, A.A.A., M.A.B. and A.-A.H.; investigation and visualization, A.A.A., M.A.B. and A.-A.H.; validation, writing—reviewing, and editing, A.A.A., M.A.B. and A.-A.H. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by King Khalid University, Grant (RGP.2/102/44) and Princess Nourah bint Abdulrahman University, Grant (PNURSP2024R337)

Data Availability Statement: The authors responsible will furnish the data utilized in this study upon request.

Acknowledgments: The authors extend their appreciation to the Deanship of Scientific Research at King Khalid University for funding this work through Research Groups Program under grant (RGP.2/102/44). The authors would like to acknowledge the Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2024R337), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Oldham, K.B.; Spanier, J. *The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order*; Academic Press: Cambridge, MA, USA, 1974.
2. Podlubny, I. *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications*; Academic Press: Cambridge, MA, USA, 1999; Volume 198.
3. Miller, K.S.; Ross, B. *An Introduction to the Fractional Calculus and Fractional Differential Equations*; John Wiley and Sons: Hoboken, NJ, USA, 1993; Volume 89.
4. Diethelm, K. *The Analysis of Fractional Differential Equations: An Application-Oriented Exposition Using Differential Operators of Caputo Type*; Springer Science and Business Media: Berlin/Heidelberg, Germany, 2010.
5. Baleanu, D.; Machado, J.A.T.; Luo, A.C.J. *Fractional Dynamics and Control*; Springer Science and Business Media: Berlin/Heidelberg, Germany, 2012.
6. Agarwal, R.P.; Ahmad, B. A survey of fractional calculus inequalities. *Appl. Math. Lett.* **2007**, *21*, 611–614.
7. Wang, G.; Lou, Y. Fractional Sobolev-Hardy inequalities and Navier-Stokes equations. *Commun. Partial. Differ. Equ.* **2013**, *38*, 1003–1045.
8. Sarikaya, M.Z.; Set, E.; Yaldiz, H.; Başak, N. Hermite–Hadamard's inequalities for fractional integrals and related fractional inequalities. *Math. Comput. Model.* **2013**, *57*, 2403–2407. [CrossRef]
9. Adil Khan, M.; Mohammad, N.; Nwaeze, E.R.; Chu, Y.-M. Quantum Hermite–Hadamard inequality by means of a Green function. *Adv. Differ. Equ.* **2020**, *2020*, 99. [CrossRef]
10. Abdeljawad, T.; Baleanu, D. Monotonicity results for fractional difference operators with discrete exponential kernels. *Adv. Differ. Equ.* **2017**, *2017*, 78. [CrossRef]
11. Rashid, S.; Abdeljawad, T.; Jarad, F.; Noor, M.A. Some Estimates for Generalized Riemann-Liouville Fractional Integrals of Exponentially Convex Functions and Their Applications. *Mathematics* **2019**, *7*, 807. [CrossRef]
12. Wang, M.; Zhang, W.; Chu, Y. Monotonicity, Convexity and Inequalities Involving the Generalized Elliptic Integrals. *Acta Math. Sci.* **2019**, *39*, 1440–1450. [CrossRef]

13. Khan, K.A.; Fatima, S.; Nosheen, A.; Matendo Mabela, R. New Developments of Hermite–Hadamard Type Inequalities via s-Convexity and Fractional Integrals. *J. Math.* **2024**, *2024*, 1997549. [CrossRef]
14. Sun, W.; Wan, H. New local fractional Hermite-Hadamard-type and Ostrowski-type inequalities with generalized Mittag-Leffler kernel for generalized h-preinvex functions. *Demonstr. Math.* **2024**, *57*, 20230128. [CrossRef]
15. Hyder, A.-A.; Barakat, M.A.; Soliman, A.H. A new class of fractional inequalities through the convexity concept and enlarged Riemann–Liouville integrals. *J. Inequalities Appl.* **2023**, *2023*, 137. [CrossRef]
16. Hyder, A.-A.; Barakat, M.A.; Fathallah, A. Enlarged integral inequalities through recent fractional generalized operators. *J. Inequalities Appl.* **2022**, *2022*, 95. [CrossRef]
17. Hyder, A.-A.; Barakat, M.A.; Fathallah, A.; Cesarano, C. Further Integral Inequalities through Some Generalized Fractional Integral Operators. *Fractal Fract.* **2021**, *5*, 282. [CrossRef]
18. Sarikaya, M.Z.; Yildirim, H. On Hermite-Hadamard Type Inequalities for Riemann-Liouville Fractional Integrals . *Miskolc Math. Notes* **2016**, *7*, 1049–1059. [CrossRef]
19. Set, E.; Choi, J.; Gözpinar, A. Hermite–Hadamard Type Inequalities for New Conformable Fractional Integral Operator, ResearchGate Preprint. 2018. Available online: https://www.researchgate.net/publication/322936389 (accessed on 8 May 2012).
20. Mihesan, V.G. A generalization of the convexity. In Proceedings of the Seminar on Functional Equations, Approximation and Convex, Cluj-Napoca, Romania, 21–25 May 1993 .
21. Hadamard, J. Etude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par Riemann. *J. Des Mathématiques Pures Appliquées* **1893**, *58*, 171–215.
22. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations, North-Holland Mathematics Studies*; Elsevier Science Publisher B.V.: Amsterdam, The Netherlands, 2006; Volume 204 .
23. Jarad, F.; Uğurlu, E.; Abdeljawad, T.; Baleanu, D. On a new class of fractional operators. *Adv. Differ. Equ.* **2017**, *2017*, 247. [CrossRef]
24. Gözpınar, A. Some Hermite-Hadamard type inequalities for convex functions via new fractional conformable integrals and related inequalities. *AIP Conf. Proc.* **2018**, *1991*, 020006.
25. Farid, G.; Rehman, A.U.; Tariq, B.; Waheed, A. On Hadamard type inequalities for m-convex functions via fractional integrals. *J. Inequal. Spec. Funct.* **2016**, *7*, 150–167.