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Existence Results and Finite-Time Stability of a Fractional (p, q)-Integro-Difference System

Mouataz Billah Mesmouli [1,*], Loredana Florentina Iambor [2,*], Amir Abdel Menaem [3] and Taher S. Hassan [1,4,5]

[1] Department of Mathematics, College of Science, University of Ha'il, Ha'il 2440, Saudi Arabia; t.hassan@uoh.edu.sa
[2] Department of Mathematics and Computer Science, University of Oradea, 410087 Oradea, Romania
[3] Department of Automated Electrical Systems, Ural Power Engineering Institute, Ural Federal University, 620002 Yekaterinburg, Russia
[4] Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt
[5] Section of Mathematics, International Telematic University Uninettuno, Corso Vittorio Emanuele II, 39, 00186 Rome, Italy
* Correspondence: m.mesmouli@uoh.edu.sa (M.B.M.); iambor.loredana@uoradea.ro (L.F.I.)

Abstract: In this article, we mainly generalize the results in the literature for a fractional q-difference equation. Our study considers a more comprehensive type of fractional (p, q)-difference system of nonlinear equations. By the Banach contraction mapping principle, we obtain a unique solution. By Krasnoselskii's fixed-point theorem, we prove the existence of solutions. In addition, finite stability has been established too. The main results in the literature have been proven to be a particular corollary of our work.

Keywords: fractional-order derivative; (p, q)-difference calculus; stability; fixed-point theorem

MSC: 34A08; 39A30; 39A13; 34K20

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1. Introduction

The aim of this study is to present and analyze a fractional hybrid (p, q)-integro-difference system

$$\begin{cases} {}^{c}\mathfrak{D}^{\lambda}_{p,q}(y(t) - f_1(t,y(t))) = Af_2\left(p^{\lambda}t,y\left(p^{\lambda}t\right)\right) + B\mathfrak{I}^{\delta}_{p,q}f_3\left(p^{\lambda+\delta}t,y\left(p^{\lambda+\delta}t\right)\right), & 0 \le t \le 1, \\ y(0) = \frac{C}{p^{\binom{\zeta}{2}}\Gamma_{p,q}(\sigma)} \int_0^{\zeta} (\zeta - qs)^{(\sigma-1)}_{p,q} y(ps) \mathrm{d}_{p,q}s, & \sigma > 0, 0 < \zeta < 1, \end{cases} \tag{1}$$

where $0 < q < p \le 1$, ${}^{c}\mathfrak{D}^{\lambda}_{p,q}$ denotes the Caputo-type fractional (p, q)-derivative of order $\lambda \in (0, 1]$ and $\mathfrak{I}^{\delta}_{p,q}$ denotes the Riemann–Liouville (p, q)-integral such that $\delta \in (0, 1)$, A, B, and C are constant matrices, and $f_1, f_2, f_3 : [0, 1] \times \mathbb{R}^n \to \mathbb{R}^n$ are given continuous functions such that $f_1(0, 0) = 0$ for simplicity.

(p, q)-difference calculus, also known as quantum calculus, is a well-established field that has demonstrated a wide range of applications in quantum mechanics, particle physics, hypergeometric series, and complex analysis. In the past few years, several studies have focused on examining the fractional problems involving q-difference or (p, q)-difference (see [1–17] and the references therein) and we suggest that readers seeking a general overview of fractional q-calculus refer to [18,19].

In [20], the authors presented the results of existence and stability for fractional hybrid q-difference equations of the form

$$\begin{cases} {}^{c}\mathfrak{D}^{\lambda}_{q}(y(t) - f_1(t,y(t))) = f_2(t,y(t)), & 0 \le t \le 1, \\ y(0) = y_0, \quad y_0 \in \mathbb{R}, 0 < \lambda \le 1. \end{cases} \tag{2}$$

After that, in 2022, Agarwal et al. [21] generalized problem (2) to the form (1) but with a one-dimensional space variable. A, B, and C are constant real numbers and $p = 1$. The authors investigated the existence, uniqueness and stability of the solution. So, in this paper, we propose a general problem in the form of (1) and we study the existence and finite-time stability with respect to the nonlocal condition.

The remaining portions of the paper are arranged as follows. In the next section, we will lay the foundation for the most important aspects related to (p,q)-difference and fractional calculus. Section 3 is devoted to establishing some criteria for the existence and uniqueness of solutions of system (1). Finite-time stability is the subject of Section 4. We give an example for our results in Section 5. Furthermore, an appropriate discussion and corollaries are provided in the last section to show the feasibility of our results.

2. Essential Materials

This part presents some essential materials which are required for our study. We begin with some fundamental definitions and results of q-calculus and (p,q)-calculus, which can be found in [1,10–12]. Let $0 < q < p \leq 1$ be constants; then, we have the following relations of (p,q)-calculus

$$[\mathfrak{m}]_{p,q} := \begin{cases} \frac{p^{\mathfrak{m}}-q^{\mathfrak{m}}}{p-q} = p^{\mathfrak{m}-1}[\mathfrak{m}]_{\frac{q}{p}}, & \mathfrak{m} \in \mathbb{N}^+, \\ 1, & \mathfrak{m} = 0, \end{cases}$$

where

$$[\mathfrak{m}]_{\frac{q}{p}} := \frac{1-\left(\frac{q}{p}\right)^{\mathfrak{m}}}{1-\frac{q}{p}},$$

$$[\mathfrak{m}]_{p,q}! := \begin{cases} [\mathfrak{m}]_{p,q}[\mathfrak{m}-1]_{p,q}....[1]_{p,q} = \prod_{i=1}^{\mathfrak{m}} \frac{p^i-q^i}{p-q}, & \mathfrak{m} \in \mathbb{N}^+, \\ 1, & \mathfrak{m} = 0. \end{cases}$$

The q-analogue of the power function $(\mathfrak{a}-\mathfrak{b})_q^{(\mathfrak{n})}$ is given by

$$(\mathfrak{a}-\mathfrak{b})_q^{(\mathfrak{n})} := \begin{cases} \prod_{i=0}^{\mathfrak{n}-1} (\mathfrak{a}-\mathfrak{b}q^i), & \mathfrak{n} \in \mathbb{N}^+,\ \mathfrak{a},\mathfrak{b} \in \mathbb{R} \\ 1, & \mathfrak{n} = 0. \end{cases}$$

The (p,q)-analogue of the power function $(\mathfrak{a}-\mathfrak{b})_{p,q}^{(\mathfrak{n})}$ is given by

$$(\mathfrak{a}-\mathfrak{b})_{p,q}^{(\mathfrak{n})} := \begin{cases} \prod_{i=0}^{\mathfrak{n}-1} (\mathfrak{a}p^i-\mathfrak{b}q^i), & \mathfrak{n} \in \mathbb{N}^+,\ \mathfrak{a},\mathfrak{b} \in \mathbb{R} \\ 1, & \mathfrak{n} = 0, \end{cases}$$

and for $\lambda \in \mathbb{R}$, the general form of the above is given by

$$(\mathfrak{a}-\mathfrak{b})_{p,q}^{(\lambda)} := p^{\binom{\lambda}{2}}(\mathfrak{a}-\mathfrak{b})_{\frac{q}{p}}^{(\lambda)} = \mathfrak{a}^{\lambda} p^{\binom{\lambda}{2}} \prod_{i=0}^{\infty} \frac{\mathfrak{a}-\mathfrak{b}\left(\frac{q}{p}\right)^i}{\mathfrak{a}-\mathfrak{b}\left(\frac{q}{p}\right)^{\lambda+i}}, \quad 0 < \mathfrak{b} < \mathfrak{a},$$

where $p^{\binom{\lambda}{2}} := \frac{\lambda(\lambda-1)}{2}$.

Definition 1 ([10])**.** *Let $0 < q < p \leq 1$. The (p,q)-derivative of the function w is defined as*

$$\mathfrak{D}_{p,q}w(t) := \frac{w(pt)-w(qt)}{(p-q)t}, \quad t \neq 0,$$

and $(\mathfrak{D}_{p,q}w)(0) = \lim_{t\to 0}(\mathfrak{D}_{p,q}w)(t)$, *such that* w *is differentiable at* 0. *Moreover, the high-order* (p,q) *-derivative* $\mathfrak{D}^n_{p,q}w(t)$ *is defined by*

$$\mathfrak{D}^n_{p,q}w(t) = \begin{cases} \mathfrak{D}_{p,q}\mathfrak{D}^{n-1}_{p,\,q}w(t), & n \in \mathbb{N}^+, \\ w(t), & n = 0. \end{cases}$$

Definition 2 ([10]). *Let* $0 < q < p \leq 1$ *and* w *be an arbitrary function of real number* t. *The* (p,q)*-integral of the function* w *is defined as*

$$\mathfrak{I}_{p,q}w(t) := \int_0^t w(s)\mathrm{d}_{p,q}s = (p-q)t\sum_{i=0}^{\infty}\frac{q^i}{p^{i+1}}w\left(\frac{q^i}{p^{i+1}}t\right),$$

such that the series on the right side converges and we call w (p,q)*-integrable on* $[0,t]$.

Definition 3 ([11]). *The* (p,q)*-Gamma function for* $\lambda \in \mathbb{R}$ *is given by*

$$\Gamma_{p,q}(\lambda) = \frac{(p-q)^{(\lambda-1)}_{p,q}}{(p-q)^{\lambda-1}},$$

with $\Gamma_{p,q}(\lambda+1) = [\lambda]_{p,q}\Gamma_{p,q}(\lambda)$.

Definition 4 ([11]). *Let* $\lambda > 0$, $0 < q < p \leq 1$, *and* $w : [0,1] \to \mathbb{R}$ *be an arbitrary function. The fractional* (p,q)*-integral of order* λ *is defined by*

$$\mathfrak{I}^{\lambda}_{p,q}w(t) = \frac{1}{p^{\binom{\lambda}{2}}\Gamma_{p,q}(\lambda)}\int_0^t (t-qs)^{(\lambda-1)}_{p,q}w\left(\frac{s}{p^{\lambda-1}}\right)\mathrm{d}_{p,q}s,$$

and $\mathfrak{I}^0_{p,q}w(t) = w(t)$.

Definition 5 ([11]). *Let* $\lambda \in (0,1]$, $0 < q < p \leq 1$, *and* w *be an arbitrary function on* $[0,1]$. *The Caputo-type fractional* (p,q)*-derivative of order* λ *is defined by*

$$\begin{aligned} {}^c\mathfrak{D}^{\lambda}_{p,q}w(t) &= \mathfrak{I}^{1-\lambda}_{p,q}\mathfrak{D}_{p,q}w(t) \\ &= \frac{1}{p^{\binom{1-\lambda}{2}}\Gamma_{p,q}(1-\lambda)}\int_0^t (t-qs)^{(-\lambda)}_{p,q}\mathfrak{D}_{p,q}w\left(\frac{s}{p^{-\lambda}}\right)\mathrm{d}_{p,q}s, \end{aligned}$$

and ${}^c\mathfrak{D}^0_{p,q}w(t) = w(t)$.

Lemma 1 ([11]). *For* $\lambda \in (\mathfrak{m}-1,\mathfrak{m}]$, $\mathfrak{m} \in \mathbb{N}$, $0 < q < p \leq 1$, *and* $w : [0,1] \to \mathbb{R}$, *we get*

$$\mathfrak{I}^{\lambda}_{p,q}\left({}^c\mathfrak{D}^{\lambda}_{p,q}w(t)\right) = w(t) - \sum_{k=0}^{\mathfrak{m}-1}\frac{t^k}{p^{\binom{\lambda}{2}}\Gamma_{p,q}(k+1)}\mathfrak{D}^k_{p,q}(0), \tag{3}$$

Indeed, for equation ${}^c\mathfrak{D}^{\lambda}_{p,q}w(t) = 0$, *the general solution is expressed by*

$$w(t) = c_0 + c_1t + c_2t^2 + \ldots + c_{\mathfrak{m}-1}t^{\mathfrak{m}-1},$$

where $c_0, c_1, c_2, \ldots, c_{\mathfrak{m}-1} \in \mathbb{R}$. *In addition,*

$${}^c\mathfrak{D}^{\lambda}_{p,q}\mathfrak{I}^{\lambda}_{p,q}w(t) = w(t). \tag{4}$$

Lemma 2. *Let, $0 < q < p \leq 1$. Then, $y(t)$ is a solution of problem (1) if and only if $y(t)$ satisfies the equation*

$$\begin{aligned} y(t) &= f_1(t,y(t)) + \frac{A}{p^{\binom{\lambda}{2}}\Gamma_{p,q}(\lambda)} \int_0^t (t-qs)_{p,q}^{(\lambda-1)} f_2(ps,y(ps))\mathrm{d}_{p,q}s \\ &\quad + \frac{B}{p^{\binom{\lambda+\delta}{2}}\Gamma_{p,q}(\lambda+\delta)} \int_0^t (t-qs)_{p,q}^{(\lambda+\delta-1)} f_3(ps,y(ps))\mathrm{d}_{p,q}s \\ &\quad + \frac{C}{p^{\binom{\sigma}{2}}\Gamma_{p,q}(\sigma)} \int_0^\zeta (\zeta-qs)_{p,q}^{(\sigma-1)} y(ps)\mathrm{d}_{p,q}s - f_1\Big(0, C\mathfrak{I}_{p,q}^{\sigma} y(\zeta)\Big). \end{aligned} \tag{5}$$

Proof. We employ the operator $\mathfrak{I}_{p,q}^{\lambda}$ on both sides of (1) by using Lemma 1 with $\mathfrak{m} = 1$ to obtain

$$y(t) - f_1(t,y(t)) = c_0 + A\mathfrak{I}_{p,q}^{\lambda} f_2\Big(p^{\lambda}t, y\Big(p^{\lambda}t\Big)\Big) + B\mathfrak{I}_{p,q}^{\lambda}\mathfrak{I}_{p,q}^{\delta} f_3\Big(p^{\lambda+\delta}t, y\Big(p^{\lambda+\delta}t\Big)\Big),$$

where $c_0 = y(0) - f_1(0, y(0))$ is a constant deduced from expansion (3). From $\lambda \in (0,1]$ and using properties of integrals, we get

$$y(t) - f_1(t,y(t)) = c_0 + A\mathfrak{I}_{p,q}^{\lambda} f_2\Big(p^{\lambda}t, y\Big(p^{\lambda}t\Big)\Big) + B\mathfrak{I}_{p,q}^{\lambda+\delta} f_3\Big(p^{\lambda+\delta}t, y\Big(p^{\lambda+\delta}t\Big)\Big),$$

and by the nonlocal condition, we obtain

$$c_0 = \frac{C}{p^{\binom{\sigma}{2}}\Gamma_{p,q}(\sigma)} \int_0^\zeta (\zeta-qs)_{p,q}^{(\sigma-1)} y(ps)\mathrm{d}_{p,q}s - f_1\Big(0, C\mathfrak{I}_{p,q}^{\sigma} y(\zeta)\Big).$$

Hence,

$$\begin{aligned} y(t) &= f_1(t,y(t)) + \frac{A}{p^{\binom{\lambda}{2}}\Gamma_{p,q}(\lambda)} \int_0^t (t-qs)_{p,q}^{(\lambda-1)} f_2(ps,y(ps))\mathrm{d}_{p,q}s \\ &\quad + \frac{B}{p^{\binom{\lambda+\delta}{2}}\Gamma_{p,q}(\lambda+\delta)} \int_0^t (t-qs)_{p,q}^{(\lambda+\delta-1)} f_3(ps,y(ps))\mathrm{d}_{p,q}s \\ &\quad + \frac{C}{p^{\binom{\sigma}{2}}\Gamma_{p,q}(\sigma)} \int_0^\zeta (\zeta-qs)_{p,q}^{(\sigma-1)} y(ps)\mathrm{d}_{p,q}s - f_1\Big(0, C\mathfrak{I}_{p,q}^{\sigma} y(\zeta)\Big). \end{aligned}$$

On the other hand, for the reverse process, we apply the Caputo fractional (p,q)-derivative of order $\lambda \in (0,1]$ on the both sides of (5) to get

$$\begin{aligned} {}^c\mathfrak{D}_{p,q}^{\lambda} y(t) &= {}^c\mathfrak{D}_{p,q}^{\lambda} f_1(t,y(t)) + A\,{}^c\mathfrak{D}_{p,q}^{\lambda}\mathfrak{I}_{p,q}^{\lambda} f_2\Big(p^{\lambda}t, y\Big(p^{\lambda}t\Big)\Big) + B\,{}^c\mathfrak{D}_{p,q}^{\lambda}\mathfrak{I}_{p,q}^{\lambda+\delta} f_3\Big(p^{\lambda+\delta}t, y\Big(p^{\lambda+\delta}t\Big)\Big) \\ &\quad + {}^c\mathfrak{D}_{p,q}^{\lambda}\left(\frac{C}{p^{\binom{\sigma}{2}}\Gamma_{p,q}(\sigma)} \int_0^\zeta (\zeta-qs)_{p,q}^{(\sigma-1)} y(ps)\mathrm{d}_{p,q}s - f_1\Big(0, C\mathfrak{I}_{p,q}^{\sigma} y(\zeta)\Big)\right), \end{aligned}$$

Then, by using (4), we get

$${}^c\mathfrak{D}_{p,q}^{\lambda}(y(t) - f_1(t,y(t))) = Af_2\Big(p^{\lambda}t, y\Big(p^{\lambda}t\Big)\Big) + B\mathfrak{I}_{p,q}^{\delta} f_3\Big(p^{\delta}t, y\Big(p^{\delta}t\Big)\Big),$$

and clearly

$$y(0) = \frac{C}{p^{\binom{\sigma}{2}}\Gamma_{p,q}(\sigma)} \int_0^\zeta (\zeta-qs)_{p,q}^{(\sigma-1)} y(ps)\mathrm{d}_{p,q}s.$$

□

3. Existence and Uniqueness of a Solution

For $y = (y_1, y_2, ..., y_n) \in \mathbb{R}^n$ and $A = (a_{ij}) \in \mathbb{R}^{n\times n}$, we denote the Euclidean vector norm $\|y\| = \sum_{i=1}^{n} |y_i|$ and the matrix norm $\|A\| = \max_{1\le j\le n} \sum_{i=1}^{n} |a_{ij}|$. Let $\mathfrak{C}$ be the Banach space of all vector-value continuous functions from $[0,1]$ to $\mathbb{R}^n$ endowed with the norm

$$\|y\|_{\mathfrak{C}} = \max_{t\in[0,1]} \|y(t)\|.$$

Lemma 2 allows us to convert problem (1) into a fixed-point problem $y = \Psi y$, where the operator $\Psi : \mathfrak{C} \to \mathfrak{C}$ is defined by

$$\begin{aligned}(\Psi y)(t) &= f_1(t,y(t)) + \frac{A}{p^{\binom{\lambda}{2}}\Gamma_{p,q}(\lambda)} \int_0^t (t-qs)_{p,q}^{(\lambda-1)} f_2(ps, y(ps)) \mathrm{d}_{p,q}s \\ &\quad + \frac{B}{p^{\binom{\lambda+\delta}{2}}\Gamma_{p,q}(\lambda+\delta)} \int_0^t (t-qs)_{p,q}^{(\lambda+\delta-1)} f_3(ps, y(ps)) \mathrm{d}_{p,q}s \\ &\quad + \frac{C}{p^{\binom{\sigma}{2}}\Gamma_{p,q}(\sigma)} \int_0^\zeta (\zeta-qs)_{p,q}^{(\sigma-1)} y(ps) \mathrm{d}_{p,q}s - f_1\Big(0, C\mathfrak{I}_{p,q}^{\sigma} y(\zeta)\Big). \end{aligned} \tag{6}$$

So, solutions of problem (1) are exclusively dependent on Ψ possessing fixed points.

The initial outcome of our investigation, which applies the Banach contraction mapping principle, addresses the existence of a unique solution to the system under consideration.

Theorem 1. *Suppose that for all $y, z \in \mathbb{R}^n$ and $t \in [0,1]$, there exist Lipschitz constants L_j, $j = 1,2,3$ such that $\|f_j(t,y) - f_j(t,z)\|_{\mathfrak{C}} \le L_j \|y - z\|_{\mathfrak{C}}$. If*

$$\Lambda := L_1 + \frac{\|A\|L_2}{\left|p^{\binom{\lambda}{2}}\right|\Gamma_{p,q}(\lambda+1)} + \frac{\|B\|L_3}{\left|p^{\binom{\lambda+\delta}{2}}\right|\Gamma_{p,q}(\lambda+\delta+1)} + \frac{\|C\|\zeta^{(\sigma)}(1+L_1)}{\left|p^{\binom{\sigma}{2}}\right|\Gamma_{p,q}(\sigma+1)} < 1, \tag{7}$$

then problem (1) possesses a unique continuous solution on the interval $[0,1]$.

Proof. In order to validate the hypotheses of the Banach contraction mapping principle, we undertake a two-step verification procedure. We set

$$K_j = \max_{t\in[0,1]} \|f_j(t,0)\| < \infty, \quad j = 1,2,3,$$

and consider the closed ball $\mathcal{B}_r = \{y \in \mathfrak{C} : \|y\|_{\mathfrak{C}} \le r\}$, where

$$K_1 + \frac{\|A\|K_2}{\left|p^{\binom{\lambda}{2}}\right|\Gamma_{p,q}(\lambda+1)} + \frac{\|B\|K_3}{\left|p^{\binom{\lambda+\delta}{2}}\right|\Gamma_{p,q}(\lambda+\delta+1)} \le (1-\Lambda)r.$$

First, we prove $\Psi\mathcal{B}_r \subset \mathcal{B}_r$. The Lipschitz conditions of the functions f_j give

$$\|f_j(t,y)\|_{\mathfrak{C}} \le \|f_j(t,y) - f_j(t,0)\|_{\mathfrak{C}} + \|f_j(t,0)\|_{\mathfrak{C}} \le L_j r + K_j, \quad j = 1,2,3.$$

Then, for $y \in \mathcal{B}_r$, $t \in [0,1]$, we get

$$
\begin{aligned}
\|(\Psi y)(t)\| \leq{}& \|f_1(t,y(t))\| + \frac{\|A\|}{\left|p^{\binom{\lambda}{2}}\right|\Gamma_{p,q}(\lambda)} \int_0^t (t-qs)_{p,q}^{(\lambda-1)} \|f_2(ps,y(ps))\| \mathrm{d}_{p,q}s \\
&+ \frac{\|B\|}{\left|p^{\binom{\lambda+\delta}{2}}\right|\Gamma_{p,q}(\lambda+\delta)} \int_0^t (t-qs)_{p,q}^{(\lambda+\delta-1)} \|f_3(ps,y(ps))\| \mathrm{d}_{p,q}s \\
&+ \frac{\|C\|}{\left|p^{\binom{\sigma}{2}}\right|\Gamma_{p,q}(\sigma)} \int_0^\zeta (\zeta-qs)_{p,q}^{(\sigma-1)} \|y(ps)\| \mathrm{d}_{p,q}s + \left\| f\left(0, C\mathfrak{I}_{p,q}^{\sigma} y(\zeta)\right)\right\| \\
&\leq L_1 r + K_1 + \frac{\|A\|(L_2 r + K_2)}{\left|p^{\binom{\lambda}{2}}\right|\Gamma_{p,q}(\lambda)} \int_0^t (t-qs)_{p,q}^{(\lambda-1)} \mathrm{d}_{p,q}s \\
&+ \frac{\|B\|(L_3 r + K_3)}{\left|p^{\binom{\lambda+\delta}{2}}\right|\Gamma_{p,q}(\lambda+\delta)} \int_0^t (t-qs)_{p,q}^{(\lambda-1)} \mathrm{d}_{p,q}s \\
&\frac{\|C\| r}{\left|p^{\binom{\sigma}{2}}\right|\Gamma_{p,q}(\sigma)} \int_0^\zeta (\zeta-qs)_{p,q}^{(\sigma-1)} \mathrm{d}_{p,q}s + \left\| f_1\left(0, C\mathfrak{I}_{p,q}^{\sigma} y(\zeta)\right)\right\| \\
&\leq \left(L_1 + \frac{\|A\| L_2}{\left|p^{\binom{\lambda}{2}}\right|\Gamma_{p,q}(\lambda+1)} + \frac{\|B\| L_3}{\left|p^{\binom{\lambda+\delta}{2}}\right|\Gamma_{p,q}(\lambda+\delta+1)} + \frac{\|C\| \zeta^{(\sigma)}(1+L_1)}{\left|p^{\binom{\sigma}{2}}\right|\Gamma_{p,q}(\sigma+1)} \right) r \\
&+ K_1 + \frac{\|A\| K_2}{\left|p^{\binom{\lambda}{2}}\right|\Gamma_{p,q}(\lambda+1)} + \frac{\|B\| K_3}{\left|p^{\binom{\lambda+\delta}{2}}\right|\Gamma_{p,q}(\lambda+\delta+1)} \\
&\leq \Lambda r + (1-\Lambda) r,
\end{aligned}
$$

and hence $\|\Psi y\|_{\mathfrak{C}} \leq r$. Then, $\Psi\mathcal{B}_r \subset \mathcal{B}_r$.

Secondly, for any $y, z \in \mathbb{R}^n$, $t \in [0,1]$, we get

$$
\begin{aligned}
&\|(\Psi y)(t) - (\Psi z)(t)\| \\
&\leq \|f_1(t,y(t)) - f_1(t,z(t))\| \\
&+ \frac{\|A\|}{\left|p^{\binom{\lambda}{2}}\right|\Gamma_{p,q}(\lambda)} \int_0^t (t-qs)_{p,q}^{(\lambda-1)} \|f_2(ps,y(ps)) - f_2(ps,z(ps))\| \mathrm{d}_{p,q}s \\
&+ \frac{\|B\|}{\left|p^{\binom{\lambda+\delta}{2}}\right|\Gamma_{p,q}(\lambda+\delta)} \int_0^t (t-qs)_{p,q}^{(\lambda+\delta-1)} \|f_3(ps,y(ps)) - f_3(ps,z(ps))\| \mathrm{d}_{p,q}s \\
&\leq L_1 \|y-z\|_{\mathfrak{C}} + \frac{\|A\| L_2 \|y-z\|_{\mathfrak{C}}}{\left|p^{\binom{\lambda}{2}}\right|\Gamma_{p,q}(\lambda)} \int_0^t (t-qs)_{p,q}^{(\lambda-1)} \mathrm{d}_{p,q}s \\
&+ \frac{\|B\| L_3 \|y-z\|_{\mathfrak{C}}}{\left|p^{\binom{\lambda+\delta}{2}}\right|\Gamma_{p,q}(\lambda+\delta)} \int_0^t (t-qs)_{p,q}^{(\lambda-1)} \mathrm{d}_{p,q}s,
\end{aligned}
$$

and hence

$$\|\Psi y - \Psi z\|_{\mathfrak{C}} \leq \Lambda \|y-z\|_{\mathfrak{C}}.$$

Since Λ is less than 1, it follows that Ψ is a contraction. Consequently, we can infer from the Banach contraction mapping principle that the operator Ψ possesses a distinct fixed point, which serves as the unique continuous solution to system (1). □

The existence result that follows is derived from Krasnoselskii's fixed-point theorem.

Lemma 3 (Krasnoselskii [22])**.** *Let Π be a nonempty, convex, closed, and bounded subset of a Banach space S. Assume that Ψ_1 and Ψ_2 map Π into S such that (i) Φ_1 is a contraction mapping* *on Π; (ii) Ψ_2 is completely continuous on Π; (iii) $s, w \in \Pi$, implies $\Psi_1 s + \Psi_2 w \in \Pi$. Then, there exists $s \in \Pi$ with $s = \Psi_1 s + \Psi_2 s$.*

Consider the set $\Pi = \{y \in \mathfrak{C} : \|y\|_{\mathfrak{C}} \leq R\}$ for an arbitrary constant R. According to Lemma 2 we define operators Ψ_1 and Ψ_2 on Π to $\mathbb{R}^n$ as

$$(\Psi_1 y)(t) = f_1(t,y(t)) + \frac{C}{p^{\binom{\sigma}{2}}\Gamma_{p,q}(\sigma)} \int_0^\zeta (\zeta - qs)_{p,q}^{(\sigma-1)} y(ps) \mathrm{d}_{p,q}s - f_1\Big(0, C\mathfrak{I}_{p,q}^{\sigma} y(\zeta)\Big), \quad (8)$$

and

$$(\Psi_2 y)(t) = \frac{A}{p^{\binom{\lambda}{2}}\Gamma_{p,q}(\lambda)} \int_0^t (t - qs)_{p,q}^{(\lambda-1)} f_2(ps, y(ps)) \mathrm{d}_{p,q}s + \frac{B}{p^{\binom{\lambda+\delta}{2}}\Gamma_{p,q}(\lambda+\delta)} \int_0^t (t - qs)_{p,q}^{(\lambda+\delta-1)} f_3(ps, y(ps)) \mathrm{d}_{p,q}s. \quad (9)$$

Theorem 2. *Suppose that for all $y, z \in \mathbb{R}^n$ and $t \in [0,1]$, there exist Lipschitz constants L_1 that satisfy $\|f_1(t,y) - f_1(t,z)\|_{\mathfrak{C}} \leq L_1 \|y - z\|_{\mathfrak{C}}$ such that*

$$L_1 + \frac{\|C\| \zeta^{(\sigma)}(1 + L_1)}{\left|p^{\binom{\sigma}{2}}\right| \Gamma_{p,q}(\sigma+1)} < 1. \quad (10)$$

If the functions $f_2(t,y)$ and $f_3(t,y)$ are bounded for all $(t,y) \in [0,1] \times \Pi$, then problem (1) possesses at least one continuous solution on the interval $[0,1]$.

Proof. Since the functions $f_2(t,y)$ and $f_3(t,y)$ are bounded for all $(t,y) \in [0,1] \times \Pi$, for any $y \in \Pi$ and $t \in [0,1]$ we denote

$$\widetilde{L_2} = \max_{(t,y)\in[0,1]\times\Pi} \|f_2(t,y)\|, \quad \widetilde{L_3} = \max_{(t,y)\in[0,1]\times\Pi} \|f_3(t,y)\|,$$

and take R such that

$$R \geq \left(K_1 + \frac{\|A\|\widetilde{L_2}}{\left|p^{\binom{\lambda}{2}}\right|\Gamma_{p,q}(\lambda+1)} + \frac{\|B\|\widetilde{L_2}}{\left|p^{\binom{\lambda+\delta}{2}}\right|\Gamma_{p,q}(\lambda+\delta+1)}\right)\left(1 - L_1 - \frac{\|C\|\zeta^{(\sigma)}(1+L_1)}{\left|p^{\binom{\sigma}{2}}\right|\Gamma_{p,q}(\sigma+1)}\right)^{-1}.$$

Step 1. For any $y, z \in \Pi$ and $t \in [0,1]$, we have

$$\begin{aligned}
\|\Psi_1 y + \Psi_2 z\|_{\mathfrak{C}} &\leq L_1 R + K_1 + \frac{\|C\|\zeta^{(\sigma)}(1+L_1)}{\left|p^{\binom{\sigma}{2}}\right|\Gamma_{p,q}(\sigma+1)} R \\
&+ \frac{\|A\|\widetilde{L_2}}{\left|p^{\binom{\lambda}{2}}\right|\Gamma_{p,q}(\lambda)} \int_0^t (t-qs)_{p,q}^{(\lambda-1)} \mathrm{d}_{p,q}s \\
&+ \frac{\|B\|\widetilde{L_3}}{\left|p^{\binom{\lambda+\delta}{2}}\right|\Gamma_{p,q}(\lambda+\delta)} \int_0^t (t-qs)_{p,q}^{(\lambda+\delta-1)} \mathrm{d}_{p,q}s \\
&\leq L_1 R + \frac{\|C\|\zeta^{(\sigma)}(1+L_1)}{\left|p^{\binom{\sigma}{2}}\right|\Gamma_{p,q}(\sigma+1)} R \\
&+ K_1 + \frac{\|A\|\widetilde{L_2}}{\left|p^{\binom{\lambda}{2}}\right|\Gamma_{p,q}(\lambda+1)} + \frac{\|B\|\widetilde{L_2}}{\left|p^{\binom{\lambda+\delta}{2}}\right|\Gamma_{p,q}(\lambda+\delta+1)} \\
&\leq R
\end{aligned}$$

Thus, $\Psi_1 y + \Psi_2 z \in \Pi$. The continuity of f_2 and f_3 implies that the operator Ψ_2 is continuous. In addition, since

$$\|\Psi_2 z\|_{\mathfrak{C}} \leq \frac{\|A\|\widetilde{L_2}}{\left|p^{\binom{\lambda}{2}}\right|\Gamma_{p,q}(\lambda)} + \frac{\|B\|\widetilde{L_2}}{\left|p^{\binom{\lambda+\delta}{2}}\right|\Gamma_{p,q}(\lambda+\delta)},$$

then Ψ_2 is uniformly bounded on Π.

Step 2. For any $z \in \Pi$ and $t_1, t_2 \in [0,1]$, we have

$$\begin{aligned}
&\|(\Psi_2 z)(t_2) - (\Psi_2 z)(t_1)\| \\
&\leq \frac{\|A\|\widetilde{L_2}}{\left|p^{\binom{\lambda}{2}}\right|\Gamma_{p,q}(\lambda)}\left|\int_0^{t_2}(t_2 - qs)_{p,q}^{(\lambda-1)}\mathrm{d}_{p,q}s - \int_0^{t_1}(t_1 - qs)_{p,q}^{(\lambda-1)}\mathrm{d}_{p,q}s\right| \\
&+ \frac{\|B\|\widetilde{L_3}}{\left|p^{\binom{\lambda+\delta}{2}}\right|\Gamma_{p,q}(\lambda+\delta)}\left|\int_0^{t_2}(t_2 - qs)_{p,q}^{(\lambda+\delta-1)}\mathrm{d}_{p,q}s - \int_0^{t_1}(t_1 - qs)_{p,q}^{(\lambda+\delta-1)}\mathrm{d}_{p,q}s\right| \\
&\leq \frac{\|A\|\widetilde{L_2}}{\left|p^{\binom{\lambda}{2}}\right|\Gamma_{p,q}(\lambda)}\left(\int_0^{t_1}\left|(t_2 - qs)_{p,q}^{(\lambda-1)} - (t_1 - qs)_{p,q}^{(\lambda-1)}\right|\mathrm{d}_{p,q}s\right. \\
&\left.+ \int_{t_1}^{t_2}(t_2 - qs)_{p,q}^{(\lambda-1)}\mathrm{d}_{p,q}s\right) \\
&+ \frac{\|B\|\widetilde{L_3}}{\left|p^{\binom{\lambda+\delta}{2}}\right|\Gamma_{p,q}(\lambda+\delta)}\left(\int_0^{t_1}\left|(t_2 - qs)_{p,q}^{(\lambda+\delta-1)} - (t_1 - qs)_{p,q}^{(\lambda+\delta-1)}\right|\mathrm{d}_{p,q}s\right. \\
&\left.+ \int_{t_1}^{t_2}(t_2 - qs)_{p,q}^{(\lambda+\delta-1)}\mathrm{d}_{p,q}s\right),
\end{aligned}$$

which infers that the last expression is independent of z, and since $t_1 \to t_2$ tends to zero, then it tends to zero. Hence, Ψ_2 is relatively compact on Π. Hence, due to the Arzelà–Ascoli Theorem, Ψ_2 is compact on Π.

Step3. For any $y, z \in \Pi$ and $t \in [0,1]$, we have

$$\|\Psi_1 y - \Psi_1 z\|_{\mathfrak{C}} \leq \left(L_1 + \frac{\|C\|\zeta^{(\sigma)}(1+L_1)}{\left|p^{\binom{\sigma}{2}}\right|\Gamma_{p,q}(\sigma+1)}\right)\|y-z\|_{\mathfrak{C}}.$$

According to condition (10), it can be concluded that Ψ_1 is a contraction. The satisfaction of the hypothesis presented in Lemma 3 allows us to guarantee the existence of at least one continuous solution for problem (1) over the interval $[0,1]$. □

4. Finite Stability of the Solutions

In this section, we introduce the stability conditions for the solutions of Equation (1).

Definition 6. *System (1) is finite-time stable with respect to the nonlocal values if for x and y, two solutions of the problem (1), there exists $\delta > 0$ such that*

$$\|x(t) - y(t)\| \leq \delta\|x(p\zeta) - y(p\zeta)\|,\ \forall t \in [0,1],\ \ 0 < p \leq 1,$$

where $0 < \zeta < 1$, related to the nonlocal condition.

Theorem 3. *Suppose that for all $y, z \in \mathbb{R}^n$ and $t \in [0,1]$, there exist Lipschitz constants $L_j, j = 1,2,3$ such that $\left\|f_j(t,y) - f_j(t,z)\right\|_{\mathfrak{C}} \leq L_j\|y-z\|_{\mathfrak{C}}$ and condition (7) holds. Then, the finite-time stability of the solution of problem (1) is achieved with respect to nonlocal values.*

Proof. Consider two solutions y_1 and y_2 of problem (1) and satisfy the fixed-point equation $\Psi y = y$, where Ψ is defined by (6). Using the stated conditions in our Theorem 3, we obtain

$$\begin{aligned}\|y_1(t) - y_2(t)\| &\le \|f_1(t,y_1(t)) - f_1(t,y_2(t))\| \\ &+ \frac{\|A\|}{\left|p^{\binom{\lambda}{2}}\right|\Gamma_{p,q}(\lambda)} \int_0^t (t-qs)_{p,q}^{(\lambda-1)} \|f_2(ps,y_1(ps)) - f_2(ps,y_2(ps))\| \mathrm{d}_{p,q}s \\ &+ \frac{\|B\|}{\left|p^{\binom{\lambda+\delta}{2}}\right|\Gamma_{p,q}(\lambda+\delta)} \int_0^t (t-qs)_{p,q}^{(\lambda+\delta-1)} \|f_3(ps,y_2(ps)) - f_3(ps,y_2(ps))\| \mathrm{d}_{p,q}s \\ &+ \|y_1(0) - y_2(0)\| + \|f_1(0,y_1(0)) - f_1(0,y_2(0))\| \\ &\le L_1\|y_1(t) - y_2(t)\| + \frac{\|A\|L_2}{\left|p^{\binom{\lambda}{2}}\right|\Gamma_{p,q}(\lambda+1)} \|y_1(t) - y_2(t)\| \\ &+ \frac{\|B\|L_3}{\left|p^{\binom{\lambda+\delta}{2}}\right|\Gamma_{p,q}(\lambda+\delta+1)} \|y_1(t) - y_2(t)\| + (1+L_1)\|y_1(0) - y_2(0)\| \\ &= (L_1 + E_1 + E_2)\|y_1(t) - y_2(t)\| + (1+L_1)\|y_1(0) - y_2(0)\|, \end{aligned} \tag{11}$$

where

$$E_1 = \frac{\|A\|L_2}{\left|p^{\binom{\lambda}{2}}\right|\Gamma_{p,q}(\lambda+1)} \text{ and } E_2 = \frac{\|B\|L_3}{\left|p^{\binom{\lambda+\delta}{2}}\right|\Gamma_{p,q}(\lambda+\delta+1)}.$$

In addition, in view of the nonlocal condition of our problem, we obtain

$$\begin{aligned}\|y_1(0) - y_2(0)\| &\le \frac{\|C\|}{\left|p^{\binom{\sigma}{2}}\right|\Gamma_{p,q}(\sigma)} \int_0^\zeta (\zeta-qs)_{p,q}^{(\sigma-1)} \|y_1(ps) - y_2(ps)\| \mathrm{d}_{p,q}s \\ &\le \frac{\|C\|}{\left|p^{\binom{\sigma}{2}}\right|\Gamma_{p,q}(\sigma)} \int_0^\zeta (\zeta-qs)_{p,q}^{(\sigma-1)} \mathrm{d}_{p,q}s \|y_1(p\zeta) - y_2(p\zeta)\|,\end{aligned}$$

Using (p,q)-integration by parts, we have

$$\begin{aligned}\left(\Im_{p,q}^{\sigma}\right)(\zeta) &= \frac{1}{\left|p^{\binom{\zeta}{2}}\right|\Gamma_{p,q}(\sigma)} \int_0^\zeta (\zeta-qs)_{p,q}^{(\sigma-1)} \mathrm{d}_{p,q}s \\ &= \frac{1}{\left|p^{\binom{\zeta}{2}}\right|\Gamma_{p,q}(\sigma)} \int_0^\zeta \frac{D_{p,q}\left((\zeta-qs)_{p,q}^{(\sigma)}\right)}{-[\sigma]_{p,q}} \mathrm{d}_{p,q}s \\ &= \frac{-1}{\left|p^{\binom{\zeta}{2}}\right|\Gamma_{p,q}(\sigma+1)} \int_0^\zeta D_{p,q}\left((\zeta-qs)_{p,q}^{(\sigma)}\right) \mathrm{d}_{p,q}s \\ &= \frac{1}{\left|p^{\binom{\zeta}{2}}\right|\Gamma_{p,q}(\sigma+1)} \zeta^{(\sigma)},\end{aligned}$$

and then

$$\|y_1(0) - y_2(0)\| \le E_3\|y_1(p\zeta) - y_2(p\zeta)\|, \tag{12}$$

where $E_3 = \frac{\|C\|\zeta^{(\sigma)}}{\left|p^{\binom{\zeta}{2}}\right|\Gamma_{p,q}(\sigma+1)}$. Substituting (12) in (11), we obtain

$$\|y_1(t) - y_2(t)\| \le \frac{E_3(1+L_1)}{1-(L_1+E_1+E_2)} \|y_1(p\zeta) - y_2(p\zeta)\|,$$

which gives the stability with respect to nonlocal values. □

5. An Example

Consider system (1) with the following quantities $y = (y_1, y_2)^T$, $\lambda = \delta = \frac{1}{4}$, $\sigma = \frac{1}{2}$, $q = 0.2$, $p = 0.25$, $\zeta = \frac{3}{4}$, and

$$A = \begin{pmatrix} \frac{1}{16} & 0 \\ \frac{1}{16} & \frac{1}{8} \end{pmatrix}, B = \begin{pmatrix} 0 & \frac{1}{12} \\ \frac{1}{6} & \frac{1}{12} \end{pmatrix}, C = \begin{pmatrix} 0 & 0 \\ \frac{1}{10} & 0 \end{pmatrix}.$$

For the functions, let

$$f_1(t,y) = \frac{t}{16}(\sin y_1, \sin y_2)^T,$$

$$f_2(t,y) = \frac{t^2}{8}(\cos y_1, \cos y_2)^T,$$

and

$$f_3(t,y) = \left(\frac{\frac{1}{7}|y_1|}{|y_1| + 1}, \tan t \right)^T.$$

Then, a simple calculation gives that $\|A\| = \frac{1}{8}$, $\|B\| = \frac{1}{6}$, $\|C\| = \frac{1}{10}$, $L_1 = \frac{1}{16}$, $L_2 = \frac{1}{8}$, $L_3 = \frac{1}{7}$, and by using the same algorithm in [23], we obtain

$$\Gamma_{p,q}(\lambda + 1) = \Gamma_{p,q}\left(\frac{5}{4}\right) = 1.4953, \text{ with } p = 0.25, q = 0.2,$$

$$\Gamma_{p,q}(\lambda + \delta + 1) = \Gamma_{p,q}(\sigma + 1) = \Gamma_{p,q}\left(\frac{3}{2}\right) = 2.2361, \text{ with } p = 0.25, q = 0.2.$$

Moreover, when we drop all the above calculations in (7), we get

$$\begin{aligned} \Lambda &= \frac{1}{16} + \frac{\frac{1}{8} \times \frac{1}{8}}{\left|\frac{\frac{1}{4}(\frac{1}{4}-1)}{2}\right| \times (1.4953)} + \frac{\frac{1}{6} \times \frac{1}{7}}{\left|\frac{\frac{1}{2}(\frac{1}{2}-1)}{2}\right| \times (2.2361)} + \frac{\frac{1}{10} \times \left(\frac{3}{4}\right)^{0.5}\left(1 + \frac{1}{16}\right)}{\left|\frac{\frac{1}{2}(\frac{1}{2}-1)}{2}\right| \times (2.2361)} \\ &\simeq 0.5883 < 1. \end{aligned}$$

As a result, by Theorem 1, the considered system has a unique solution on $[0, 1]$. On the other hand, since the conditions of Theorem 3 hold, we conclude that the solution of this example is finite-time stable.

6. Discussion and Corollaries

In our study, we took a more general problem in contrast to [20,21]. Therefore, if in our system (1) $p = 1$, $A = 1$, $B = 0$, and $C = y_0 \in \mathbb{R}$, we obtain the same results in [20]. On the other hand, if $p = 1$, A, B, and C are constant real numbers, we will reach the same results as in reference [21].

As a particular results of our work, if we take $p = 1$, system (1) will be as follows

$$\begin{cases} {}^c\mathfrak{D}_q^\lambda(y(t) - f_1(t,y(t))) = Af_2(t,y(t)) + B\mathfrak{I}_q^\delta f_3(t,y(t)), & 0 \le t \le 1, \\ y(0) = \frac{C}{\Gamma_q(\sigma)} \int_0^\zeta (\zeta - qs)_q^{(\sigma-1)} y(s) \mathrm{d}_q s, & \sigma > 0, 0 < \zeta < 1, \end{cases} \tag{13}$$

and we obtain the following corollaries

Corollary 1. *Suppose that for all $y, z \in \mathbb{R}^n$ and $t \in [0, 1]$, there exist Lipschitz constants $L_j, j = 1, 2, 3$ such that $\left\|f_j(t,y) - f_j(t,z)\right\|_{\mathfrak{C}} \le L_j\|y - z\|_{\mathfrak{C}}$. If*

$$\Lambda := L_1 + \frac{\|A\|L_2}{\Gamma_q(\lambda + 1)} + \frac{\|B\|L_3}{\Gamma_q(\lambda + \delta + 1)} + \frac{\|C\|\zeta^{(\sigma)}(1 + L_1)}{\Gamma_q(\sigma + 1)} < 1, \tag{14}$$

then problem (13) possesses a unique continuous solution on the interval $[0,1]$.

Corollary 2. *Suppose that for all* $y,z \in \mathbb{R}^n$ *and* $t \in [0,1]$, *there exist Lipschitz constants* L_1 *that satisfy* $\|f_1(t,y) - f_1(t,z)\|_{\mathfrak{C}} \leq L_1\|y - z\|_{\mathfrak{C}}$ *such that*

$$L_1 + \frac{\|C\|\zeta^{(\sigma)}(1+L_1)}{\Gamma_q(\sigma+1)} < 1. \tag{15}$$

If the functions $f_2(t,y)$ *and* $f_3(t,y)$ *are bounded for all* $(t,y) \in [0,1] \times \Pi$, *then problem (13) possesses at least one continuous solution on the interval* $[0,1]$.

Corollary 3. *Suppose that for all* $y,z \in \mathbb{R}^n$ *and* $t \in [0,1]$, *there exist Lipschitz constants* $L_j, j = 1,2,3$ *such that* $\|f_j(t,y) - f_j(t,z)\|_{\mathfrak{C}} \leq L_j\|y - z\|_{\mathfrak{C}}$ *and condition (7) holds. Then, the stability of the solution of problem (13) is achieved with respect to nonlocal values.*

Remark 1. *Note that Corollaries 1–3 are results concerned with a system of the form in (13), and are more general than those in [21], which concern the problem in one dimension.*

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