mathematics

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A New Approach of Complex Fuzzy Ideals in BCK/BCI-Algebras

Manivannan Balamurugan [1,*,†], Thukkaraman Ramesh [1,2,†], Anas Al-Masarwah [3] and Kholood Alsager [4,*]

[1] Department of Mathematics, Vel Tech Rangarajan Dr. Sagunthala R&D Institute of Science and Technology, Chennai 600 062, India
[2] Department of Mathematics, Sri Vidya Mandir Arts & Science College (Autonomous), Uthangarai 636 902, India; vtd1269@veltech.edu.in
[3] Department of Mathematics, Faculty of Science, Ajloun National University, P.O. Box 43, Ajloun 26810, Jordan; anas.almasarwah@anu.edu.jo
[4] Department of Mathematics, College of Science, Qassim University, Buraydah 52571, Saudi Arabia
* Correspondence: drbalamuruganm@veltech.edu.in (M.B.); ksakr@qu.edu.sa (K.A.)
† These authors contributed equally to this work.

Abstract: The concept of complex fuzzy sets, where the unit disk of the complex plane acts as the codomain of the membership function, as an extension of fuzzy sets. The objective of this article is to use complex fuzzy sets in BCK/BCI-algebras. We present the concept of a complex fuzzy subalgebra in a BCK/BCI-algebra and explore their properties. Furthermore, we discuss the modal and level operators of these complex fuzzy subalgebras, highlighting their importance in BCK/BCI-algebras. We study various operations, and the laws of a complex fuzzy system, including union, intersection, complement, simple differences, and bounded differences of complex fuzzy ideals within BCK/BCI-algebras. Finally, we generate a computer programming algorithm that implements our complex fuzzy subalgebras/ideals in BCK/BCI-algebras procedure for ease of lengthy calculations.

Keywords: BCK/BCI-algebras; fuzzy logic; complex fuzzy set; complex fuzzy subalgebra; complex fuzzy ideal

MSC: 06F35; 03B52; 03B05; 03B47

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Citation: Balamurugan, M.; Ramesh, T.; Al-Masarwah, A.; Alsager, K. A New Approach of Complex Fuzzy Ideals in BCK/BCI-Algebras. *Mathematics* **2024**, *12*, 1583. https://doi.org/10.3390/math12101583

Academic Editors: Cheung Chieh Ku and Chang-Hua Lien

Received: 16 April 2024
Revised: 13 May 2024
Accepted: 16 May 2024
Published: 18 May 2024

1. Introduction

The concept of fuzzy sets, initially proposed by Zadeh [1], has been used extensively by numerous researchers to address problems of uncertainty and inaccuracy in current challenges. Axiom systems for propositional calculi were developed by Imai et al. [2,3]. They consist of axioms and rules of reasoning that help people draw conclusions and show that logical arguments are valid. Theoretical logic refers to propositional logic in statements are manipulated and analyzed using logical operators and their truth values such as AND, OR, and NOT. Many mathematical systems, including propositional logic, can be modeled and analyzed using algebraic structures. In 1966, Iséki [4,5] proposed BCK/BCI-algebras as an extension of the concepts of set–theoretic difference and propositional calculus. A comprehensive analysis of the theory of BCK/BCI-algebras was subsequently published, highlighting in particular the ideal theory of BCK/BCI-algebras. Meng [6] introduced the concept of ideals of BCK-algebras.

Al-Masarwah et al. [7] studied doubt bipolar fuzzy H-ideals in BCK/BCI-algebras and investigated many interesting properties of ideals. Balamurugan et al. [8–10] introduced a polar fuzzy set as an extension of the fuzzy sets applied to BCK/BCI-algebras. Al-shami et al. [11–15] proposed the ideas of (2,1)-fuzzy sets, (a,b)-fuzzy sets, (m,n)-fuzzy sets, SR-fuzzy sets, and k_m^n-rung picture fuzzy sets, which can help people make decisions that take more than one factor into account. The idea of complex fuzzy set was first introduced by Ramot et al. [16,17]. The set of complex numbers is an extension of the set of real numbers, first introduced by Gauss in 1795. A complex fuzzy set is an extension of a fuzzy set whose range is extended from [0,1] to a disk with a radius of 1 in a complex plane. Hu et al. [18] represented the signals as complex fuzzy sets and examined their orthogonal properties. Song et al. [19], distance measures are used to quantify the dissimilarity or similarity between two data points in a given data set.

Garg et al. [20] introduced the complex intuitionistic fuzzy soft SWARA-COPRAS approach, a methodology for decision making, particularly in the context of selecting enterprise resource planning (ERP) software. It integrates various techniques such as SWARA and COPRAS within the framework of complex intuitionistic fuzzy soft sets to deal with uncertainty and ambiguity in decision-making processes. Dai [21] developed quasi-MV-algebras as mathematical constructs used in complex fuzzy logic. These structures are based on MV-algebras, which in turn generalize Boolean algebras to effectively manage complicated fuzzy inferences. This extension enables a more fine-grained and adaptive approach to modeling uncertainty and indeterminacy. Xu et al. [22] introduced complex-valued migration into complex fuzzy operations.

Yang et al. [23] introduced the complex intuitionistic fuzzy-ordered weighted distance measure, a method for evaluating the similarity or dissimilarity between complex intuitionistic fuzzy sets considering weighted attributes and their distances. Zeeshan [24] developed complex fuzzy sets with applications to decision problems. Abuhijleh et al. [25] studied complex fuzzy subgroups and their properties. Dai [26] studied complex linguistic fuzzy sets, which are concepts of fuzzy logic and fuzzy set theory that focus on thinking and decision-making in uncertain situations.

Yasin [27] introduced trigonometric similarity measures of complex fuzzy sets, which refers to mathematical methods used to quantify the similarity or dissimilarity between complex fuzzy sets. Al Tahan et al. [28–30] examined in detail the properties of complex fuzzy Hv subgroups, Krasner hyperrings and the linear Diophantine fuzzy n-fold weak subalgebra of a BE-algebra.

Motivation and objectives of the proposed method are as follows:

1.1. Motivation

1. The complex fuzzy ideals offer a natural extension of classical ideals to BCK/BCI-algebras, allowing for a more comprehensive analysis of their properties and behavior.
2. It provides a means to quantify this fuzziness within the framework of BCK/BCI-algebras.
3. It offers a versatile representation that can capture a wide range of algebraic structures and properties within BCK/BCI-algebras.
4. Complex fuzzy ideals play a crucial role in these applications by providing a formalism for reasoning about fuzzy sets, approximate reasoning, and uncertainty management within the framework of BCK/BCI-algebras.

1.2. Objectives of the Proposed Method

1. The complex fuzzy sets are a new concept that extends traditional fuzzy sets by using complex numbers or fuzzy numbers as elements.
2. We aim to apply complex fuzzy sets in BCK/BCI-algebras.
3. We present the idea of complex fuzzy subalgebras in BCK/BCI-algebras and investigate their properties.
4. We explore modal and level operators associated with complex fuzzy subalgebras in BCK/BCI-algebras.
5. We investigate various operations, such as union, intersection, complement, simple difference, and bounded difference, defined on complex fuzzy ideals within BCK/BCI-algebras.
6. We provide a systematic method for dealing with complex fuzzy sets and operations in BCK/BCI-algebras, potentially providing practical applications and computational implementations.

The structure of this research article is outlined as follows: Section 2 introduces the concepts of BCK-algebras, fuzzy subalgebras, fuzzy ideals, and complex fuzzy sets. In Section 3, a framework for complex fuzzy subalgebras within BCK/BCI-algebras is proposed, and their properties are examined. Section 4 presents various operations defined on complex fuzzy ideals, including unions, intersections, complements, simple differences, and bounded differences. Section 5 deals with a theoretical comparative analysis of the proposed approach. In Section 6, the advantages of the proposed approach are presented. At the end, Section 7 offers conclusions and outlines future research directions. In Appendix A, we generates a computer programming algorithm that implements our complex fuzzy subalgebras/ideals in BCK/BCI-algebras procedure for ease of lengthy calculations.

In this article we often use different symbols and their corresponding meanings. These symbols and their explanations are summarized in Table 1 below:

Table 1. List of symbols and abbreviations.

Symbols	Abbreviations
U	BCK/BCI-algebra
$\mathcal{CFI}$	Complex Fuzzy Ideal
$\mathcal{CFS}$	Complex Fuzzy Set
$\mathcal{CFSA}$	Complex Fuzzy Subalgebra
$\mathcal{FI}$	Fuzzy Ideal
$\mathcal{FS}$	Fuzzy Set
$\mathcal{FSA}$	Fuzzy Subalgebra

2. Preliminaries

An important class of legitimate algebras known as BCK/BCI-algebras, initially developed by Iséki (refer to [2,3]), has undergone extensive investigation by numerous researchers. We recall the concepts and basic insights necessary for this work. If the criteria are satisfied and a fixed U possesses a designated element denoted as 0 along with a binary operation "$\diamond$"

(I_1) $(\forall \check{\varsigma}, \check{\varrho}, \check{\kappa} \in U)(((\check{\varsigma} \diamond \check{\varrho}) \diamond (\check{\varsigma} \diamond \check{\kappa})) \diamond (\check{\kappa} \diamond \check{\varrho}) = 0)$,
(I_2) $(\forall \check{\varsigma}, \check{\varrho} \in U)\ ((\check{\varsigma} \diamond (\check{\varsigma} \diamond \check{\varrho})) \diamond \check{\varrho} = 0)$,
(I_3) $(\forall \check{\varsigma} \in U)(\check{\varsigma} \diamond \check{\varsigma} = 0)$,
(I_4) $(\forall \check{\varsigma}, \check{\varrho} \in U)(\check{\varsigma} \diamond \check{\varrho} = 0, \check{\varrho} \diamond \check{\varsigma} = 0 \Rightarrow \check{\varsigma} = \check{\varrho})$,

then we categorize U as a BCI-*algebra*. Furthermore, if a BCI-algebra U additionally satisfies:

(I_5) $(\forall \check{\varsigma} \in U)(0 \diamond \check{\varsigma} = 0)$, then U as a BCK-*algebra*.

A fuzzy set ($\mathcal{FS}$) $\tilde{Ą}$ defined on U is given by

$$\tilde{Ą} = \{(\check{\varsigma}, \bar{v}_{\tilde{Ą}}(\check{\varsigma}))\ :\ \check{\varsigma} \in U\},$$

where $\bar{v}_{\tilde{Ą}}\colon U \to [0,1]$ is a real-valued membership function such that $\bar{v}_{\tilde{Ą}}(\check{\varsigma}) \leq 1$, for all $\check{\varsigma} \in U$.

A $\mathcal{FS}$ $\tilde{Ą}$ of U is called a $\mathcal{FSA}$ of U if it meets

$$(\forall \check{\varsigma}, \check{\varrho} \in U, \bar{v}_{\tilde{Ą}}(\check{\varsigma} \diamond \check{\varrho}) \geq \bar{v}_{\tilde{Ą}}(\check{\varsigma}) \wedge \bar{v}_{\tilde{Ą}}(\check{\varrho})).$$

A $\mathcal{FS}$ $\tilde{Ą}$ of U is called a $\mathcal{FI}$ of U if it meets

$$(\forall \check{\varsigma}, \check{\varrho} \in U, \bar{v}_{\tilde{Ą}}(\check{\varsigma}) \geq \bar{v}_{\tilde{Ą}}(\check{\varsigma} \diamond \check{\varrho}) \wedge \bar{v}_{\tilde{Ą}}(\check{\varrho})).$$

Definition 1. *Let $\tilde{Ą}$ be a fuzzy set of U. Then, modal operators (i), (ii), and level operators (iii), (iv) are defined by*

(i). $\boxplus\tilde{A} = \{(\check{\zeta}, \frac{\bar{\upsilon}_{\tilde{A}}(\check{\zeta})}{2}) : \check{\zeta} \in U\}$,
(ii). $\boxtimes\tilde{A} = \{(\check{\zeta}, \frac{\bar{\upsilon}_{\tilde{A}}(\check{\zeta})+1}{2}) : \check{\zeta} \in U\}$,
(iii). $!\tilde{A} = \{(\check{\zeta}, \frac{1}{2} \vee \bar{\upsilon}_{\tilde{A}}(\check{\zeta})) : \check{\zeta} \in U\}$,
(iv). $?\tilde{A} = \{(\check{\zeta}, \frac{1}{2} \wedge \bar{\upsilon}_{\tilde{A}}(\check{\zeta})) : \check{\zeta} \in U\}$.

Ramot et al. [16,17] extended fuzzy set theory by introducing the concept of complex fuzzy set ($\mathcal{CFS}$) and incorporating phase angle into the analysis. They provided the following definition:

Definition 2 ([16,28])**.** *A $\mathcal{CFS}$, defined on U is characterized by the membership function $\bar{\upsilon}_{\tilde{A}}(\check{\zeta})$ that assigns any element, a complex-valued grade of membership in $\tilde{A}$. The $\mathcal{CFS}$ may be represented by the set of ordered pairs*

$$\tilde{A} = \{(\check{\zeta}, \bar{\upsilon}_{\tilde{A}}(\check{\zeta})) : \check{\zeta} \in U\},$$

where $\bar{\upsilon}_{\tilde{A}}(\check{\zeta}) = \gamma_{\tilde{A}}(\check{\zeta})e^{i\vartheta_{\tilde{A}}(\check{\zeta})}$, $i = \sqrt{-1}$, $\gamma_{\tilde{A}}(\check{\zeta}) \in [0,1]$ and $\vartheta_{\tilde{A}}(\check{\zeta}) \in [0, 2\pi]$.

Definition 3 ([28])**.** *Let $\tilde{A} = \{(\check{\zeta}, \bar{\upsilon}_{\tilde{A}}(\check{\zeta})) : \check{\zeta} \in U\}$ and $\tilde{B} = \{(\check{\zeta}, \bar{\upsilon}_{\tilde{B}}(\check{\zeta})) : \check{\zeta} \in U\}$ be complex subsets of a non-void set U with membership functions $\bar{\upsilon}_{\tilde{A}}(\check{\zeta}) = \gamma_{\tilde{A}}(\check{\zeta})e^{i\vartheta_{\tilde{A}}(\check{\zeta})}$ and $\bar{\upsilon}_{\tilde{B}}(\check{\zeta}) = \gamma_{\tilde{B}}(\check{\zeta})e^{i\vartheta_{\tilde{B}}(\check{\zeta})}$ respectively. By $\bar{\upsilon}_{\tilde{A}}(\check{\zeta}) \leq \bar{\upsilon}_{\tilde{B}}(\check{\zeta})$, we mean that $\gamma_{\tilde{A}}(\check{\zeta}) \leq \gamma_{\tilde{B}}(\check{\zeta})$ and $\vartheta_{\tilde{A}}(\check{\zeta}) \leq \vartheta_{\tilde{B}}(\check{\zeta})$.*

3. Complex Fuzzy Subalgebras of BCK/BCI-Algebras (CFSAs)

In this section, we will review basic ideas about $\mathcal{CFS}$s, and $\mathcal{CFSA}$s on the universal set $U \neq \emptyset$.

Definition 4. *A $\mathcal{CFS}$ $\tilde{A} = (\check{\zeta}, \bar{\upsilon}_{\tilde{A}}(\check{\zeta}))$ is considered a $\mathcal{CFSA}$ of U if $\check{\zeta}, \check{\varrho} \in U$, and its satisfies following:*

1. $\bar{\upsilon}_{\tilde{A}}(0) \geq \bar{\upsilon}_{\tilde{A}}(\check{\zeta})$
2. $\bar{\upsilon}_{\tilde{A}}(\check{\zeta} \diamond \check{\varrho}) \geq \bar{\upsilon}_{\tilde{A}}(\check{\zeta}) \wedge \bar{\upsilon}_{\tilde{A}}(\check{\varrho})$.

Example 1. *Take a BCK-algebra $U = \{0, \check{\zeta}, \check{\varrho}, \check{\kappa}\}$ with Table 2. Now define a $\mathcal{CFS}$ $\tilde{A}$ on U as:*

$$\tilde{A} = \{(0, 0.8e^{i0.5\pi}), (\check{\zeta}, 0.8e^{i0.5\pi}), (\check{\varrho}, 0.5e^{i0.2\pi}), (\check{\kappa}, 0.8e^{i0.5\pi})\}.$$

It is easy to show that $\tilde{A}$ is a $\mathcal{CFSA}$ of U.

Table 2. Cayley's table representing the binary operation denoted by "$\diamond$".

$\diamond$	0	$\check{\zeta}$	$\check{\varrho}$	$\check{\kappa}$
0	0	0	0	0
$\check{\zeta}$	$\check{\zeta}$	0	0	$\check{\zeta}$
$\check{\varrho}$	$\check{\varrho}$	$\check{\varrho}$	0	$\check{\varrho}$
$\check{\kappa}$	$\check{\kappa}$	$\check{\kappa}$	$\check{\kappa}$	0

Example 2. *Take a BCI-algebra $U = \{0, \check{\zeta}, \check{\varrho}, \check{\kappa}, \check{\varkappa}\}$ with Table 3. Now define a $\mathcal{CFS}$ $\tilde{A}$ on U as*

$$\tilde{A} = \{(0, 0.9e^{i0.6\pi}), (\check{\zeta}, 0.7e^{i0.5\pi}), (\check{\varrho}, 0.4e^{i0.3\pi}), (\check{\kappa}, 0.4e^{i0.3\pi}), (\check{\varkappa}, 0.4e^{i0.3\pi})\}.$$

It is easy to show that $\tilde{A}$ is a $\mathcal{CFSA}$ of U.

Table 3. Cayley's table representing the binary operation denoted by "$\diamond$".

$\diamond$	**0**	$\check{\varsigma}$	$\check{\varrho}$	$\check{\kappa}$	$\check{\varkappa}$
0	0	0	$\check{\varkappa}$	$\check{\kappa}$	$\check{\varrho}$
$\check{\varsigma}$	$\check{\varsigma}$	0	$\check{\varkappa}$	$\check{\kappa}$	$\check{\varrho}$
$\check{\varrho}$	$\check{\varrho}$	$\check{\varrho}$	0	$\check{\varrho}$	$\check{\kappa}$
$\check{\kappa}$	$\check{\kappa}$	$\check{\kappa}$	$\check{\varrho}$	0	$\check{\varkappa}$
$\check{\varkappa}$	$\check{\varkappa}$	$\check{\varkappa}$	$\check{\kappa}$	$\check{\varrho}$	0

Property 1. *If $\tilde{A}$ is a $\mathcal{CFSA}$ of U, then $\bar{\upsilon}_{\tilde{A}}(0) \geq \bar{\upsilon}_{\tilde{A}}(\check{\varsigma})$.*

Proof. Let $\tilde{A}$ be a $\mathcal{CFSA}$ of U. Then

$$\begin{aligned}
\bar{\upsilon}_{\tilde{A}}(0) &= \gamma_{\tilde{A}}(0)e^{i\vartheta_{\tilde{A}}(0)} \\
&= \gamma_{\tilde{A}}(\check{\varsigma} \diamond \check{\varsigma})e^{i\vartheta_{\tilde{A}}(\check{\varsigma}\diamond\check{\varsigma})} \\
&\geq (\gamma_{\tilde{A}}(\check{\varsigma}) \wedge \gamma_{\tilde{A}}(\check{\varsigma}))e^{i(\vartheta_{\tilde{A}}(\check{\varsigma})\wedge\vartheta_{\tilde{A}}(\check{\varsigma}))} \\
&= \gamma_{\tilde{A}}(\check{\varsigma})e^{i\vartheta_{\tilde{A}}(\check{\varsigma})} \\
&\geq \bar{\upsilon}_{\tilde{A}}(\check{\varsigma}).
\end{aligned}$$

□

Definition 5. *Let $\tilde{A}$ be a $\mathcal{CFS}$ of U. Then modal operator $\boxplus\tilde{A}$ is defined as*

$$\bar{\upsilon}_{\boxplus\tilde{A}}(\check{\varsigma}) = \frac{\gamma_{\tilde{A}}(\check{\varsigma})}{2}\, e^{i\left(\frac{\vartheta_{\tilde{A}}(\check{\varsigma})}{2}\right)}$$

Example 3. *Let $\bar{\upsilon}_{\tilde{A}}(\check{\varsigma}) = \{(\check{\varsigma}_1, 0.4e^{i0.5\pi}), (\check{\varsigma}_2, 0.8e^{i0.1\pi}), (\check{\varsigma}_3, 0.6e^{i0.3\pi})\}$ be a $\mathcal{CFS}$ of U. Then $\bar{\upsilon}_{\boxplus\tilde{A}}(\check{\varsigma}) = \{(\check{\varsigma}_1, 0.2e^{i0.25\pi}), (\check{\varsigma}_2, 0.4e^{i0.05\pi}), (\check{\varsigma}_3, 0.3e^{i0.15\pi})\}$ is a $\mathcal{CFS}$ of U.*

Theorem 1. *Let $\tilde{A}$ be a $\mathcal{CFSA}$ of U. Then $\boxplus\tilde{A}$ is a $\mathcal{CFSA}$ of U.*

Proof. For each $\check{\varsigma}, \in U$, we have

$$\begin{aligned}
\bar{\upsilon}_{\boxplus\tilde{A}}(0) &= \frac{\gamma_{\tilde{A}}(0)}{2}\, e^{i\left(\frac{\vartheta_{\tilde{A}}(0)}{2}\right)} \\
&\geq \frac{\gamma_{\tilde{A}}(\check{\varsigma})}{2}\, e^{i\left(\frac{\vartheta_{\tilde{A}}(\check{\varsigma})}{2}\right)} \\
&= \bar{\upsilon}_{\boxplus\tilde{A}}(\check{\varsigma}).
\end{aligned}$$

Let $\check{\zeta}, \check{\varrho} \in U$. Then

$$\begin{aligned}
\bar{\upsilon}_{\boxplus\tilde{\c{A}}}(\check{\zeta} \diamond \check{\varrho}) &= \frac{\gamma_{\tilde{\c{A}}}(\check{\zeta} \diamond \check{\varrho})}{2}\, e^{i\left(\frac{\vartheta_{\tilde{\c{A}}}(\check{\zeta}\diamond\check{\varrho})}{2}\right)} \\
&\geq \left(\frac{\gamma_{\tilde{\c{A}}}(\check{\zeta})}{2} \wedge \frac{\gamma_{\tilde{\c{A}}}(\check{\varrho})}{2}\right) e^{i\left(\frac{\vartheta_{\tilde{\c{A}}}(\check{\zeta})}{2} \wedge \frac{\vartheta_{\tilde{\c{A}}}(\check{\varrho})}{2}\right)} \\
&= \bar{\upsilon}_{\boxplus\tilde{\c{A}}}(\check{\zeta}) \wedge \bar{\upsilon}_{\boxplus\tilde{\c{A}}}(\check{\varrho}).
\end{aligned}$$

□

Definition 6. *Let $\tilde{\c{A}}$ be a $\mathcal{CFS}$ of U. Then modal operator $\boxtimes\tilde{\c{A}}$ is defined as*

$$\bar{\upsilon}_{\boxtimes\tilde{\c{A}}}(\check{\zeta}) = \frac{\gamma_{\tilde{\c{A}}}(\check{\zeta})+1}{2}\, e^{i\left(\frac{\vartheta_{\tilde{\c{A}}}(\check{\zeta})+1}{2}\right)}$$

Example 4. *Let $\bar{\upsilon}_{\tilde{\c{A}}}(\check{\zeta}) = \{(\check{\zeta}_1, 0.3e^{i0.5\pi}), (\check{\zeta}_2, 0.7e^{i0.2\pi}), (\check{\zeta}_3, 0.5e^{i0.4\pi})\}$ be a $\mathcal{CFS}$ of U. Then $\bar{\upsilon}_{\boxtimes\tilde{\c{A}}}(\check{\zeta}) = \{(\check{\zeta}_1, 0.8e^{i0.75\pi}), (\check{\zeta}_2, 0.85e^{i0.6\pi}), (\check{\zeta}_3, 0.75e^{i0.7\pi})\}$ is a $\mathcal{CFS}$ of U.*

Theorem 2. *Let $\tilde{\c{A}}$ be a $\mathcal{CFSA}$ of U. Then $\boxtimes\tilde{\c{A}}$ is a $\mathcal{CFSA}$ of U.*

Proof. For each $\check{\zeta}, \in U$, we have

$$\begin{aligned}
\bar{\upsilon}_{\boxtimes\tilde{\c{A}}}(0) &= \frac{\gamma_{\tilde{\c{A}}}(0)+1}{2}\, e^{i\left(\frac{\vartheta_{\tilde{\c{A}}}(0)+1}{2}\right)} \\
&\geq \frac{\gamma_{\tilde{\c{A}}}(\check{\zeta})+1}{2}\, e^{i\left(\frac{\vartheta_{\tilde{\c{A}}}(\check{\zeta})+1}{2}\right)} \\
&= \bar{\upsilon}_{\boxtimes\tilde{\c{A}}}(\check{\zeta}).
\end{aligned}$$

Let $\check{\zeta}, \check{\varrho} \in U$. Then

$$\begin{aligned}
\bar{\upsilon}_{\boxtimes\tilde{\c{A}}}(\check{\zeta} \diamond \check{\varrho}) &= \frac{\gamma_{\tilde{\c{A}}}(\check{\zeta} \diamond \check{\varrho})+1}{2}\, e^{i\left(\frac{\vartheta_{\tilde{\c{A}}}(\check{\zeta}\diamond\check{\varrho})+1}{2}\right)} \\
&\geq \left(\frac{\gamma_{\tilde{\c{A}}}(\check{\zeta})+1}{2} \wedge \frac{\gamma_{\tilde{\c{A}}}(\check{\varrho})+1}{2}\right) e^{i\left(\frac{\vartheta_{\tilde{\c{A}}}(\check{\zeta})+1}{2} \wedge \frac{\vartheta_{\tilde{\c{A}}}(\check{\varrho})+1}{2}\right)} \\
&= \bar{\upsilon}_{\boxtimes\tilde{\c{A}}}(\check{\zeta}) \wedge \bar{\upsilon}_{\boxtimes\tilde{\c{A}}}(\check{\varrho}).
\end{aligned}$$

Therefore, $\boxtimes\tilde{\c{A}}$ is a $\mathcal{CFSA}$ of U. □

Definition 7. *Let $\tilde{\c{A}}$ be a $\mathcal{CFS}$ of U. Then level operator $!\tilde{\c{A}}$ is defined as*

$$\bar{\upsilon}_{!\tilde{\c{A}}}(\check{\zeta}) = \left(\frac{1}{2} \vee \gamma_{\tilde{\c{A}}}(\check{\zeta})\right) e^{i\left(\frac{1}{2} \vee \vartheta_{\tilde{\c{A}}}(\check{\zeta})\right)}$$

Example 5. *Let $\bar{\upsilon}_{\tilde{\c{A}}}(\check{\zeta}) = \{(\check{\zeta}_1, 0.4e^{i0.6\pi}), (\check{\zeta}_2, 0.6e^{i0.2\pi}), (\check{\zeta}_3, 0.5e^{i0.7\pi})\}$ be a $\mathcal{CFS}$ of U. Then $\bar{\upsilon}_{!\tilde{\c{A}}}(\check{\zeta}) = \{(\check{\zeta}_1, 0.5e^{i0.6\pi}), (\check{\zeta}_2, 0.6e^{i0.5\pi}), (\check{\zeta}_3, 0.5e^{i0.7\pi})\}$ be a $\mathcal{CFS}$ of U*

Theorem 3. *Let $\tilde{\c{A}}$ be a $\mathcal{CFSA}$ of U. Then $!\tilde{\c{A}}$ is a $\mathcal{CFSA}$ of U.*

Proof. For each $\check{\zeta}, \in U$, we have

$$\begin{aligned}
\bar{v}_{!\tilde{A}}(0) &= \left(\frac{1}{2} \vee \gamma_{\tilde{A}}(0)\right) e^{i\left(\frac{1}{2} \vee \vartheta_{\tilde{A}}(0)\right)} \\
&\geq \left(\frac{1}{2} \vee \gamma_{\tilde{A}}(\check{\zeta})\right) e^{i\left(\frac{1}{2} \vee \vartheta_{\tilde{A}}(\check{\zeta})\right)} \\
&= \bar{v}_{!\tilde{A}}(\check{\zeta}).
\end{aligned}$$

Let $\check{\zeta}, \check{\varrho} \in U$. Then

$$\begin{aligned}
\bar{v}_{!\tilde{A}}(\check{\zeta} \diamond \check{\varrho}) &= \frac{1}{2} \vee \gamma_{\tilde{A}}(\check{\zeta} \diamond \check{\varrho})\, e^{i\left(\vartheta_{\tilde{A}}(\check{\zeta} \diamond \check{\varrho})\right)} \\
&\geq \frac{1}{2} \vee \left(\gamma_{\tilde{A}}(\check{\zeta}) \wedge \gamma_{\tilde{A}}(\check{\varrho})\right) e^{i\left(\vartheta_{\tilde{A}}(\check{\zeta}) \wedge \vartheta_{\tilde{A}}(\check{\varrho})\right)} \\
&= \left(\frac{1}{2} \vee \gamma_{\tilde{A}}(\check{\zeta})\, e^{i\vartheta_{\tilde{A}}(\check{\zeta})}\right) \wedge \left(\frac{1}{2} \vee \gamma_{\tilde{A}}(\check{\varrho})\, e^{i\vartheta_{\tilde{A}}(\check{\varrho})}\right) \\
&= \bar{v}_{!\tilde{A}}(\check{\zeta}) \wedge \bar{v}_{!\tilde{A}}(\check{\varrho}).
\end{aligned}$$

Therefore, $!\tilde{A}$ is a $\mathcal{CFSA}$ of U. □

Definition 8. *Let $\tilde{A}$ be a $\mathcal{CFS}$ of U. Then level operator $?\tilde{A}$ is defined as*

$$\bar{v}_{?\tilde{A}}(\check{\zeta}) = \left(\tfrac{1}{2} \wedge \gamma_{\tilde{A}}(\check{\zeta})\right) e^{i\left(\frac{1}{2} \wedge \gamma_{\tilde{A}}(\check{\vartheta})\right)}$$

Example 6. *Let $\bar{v}_{\tilde{A}}(\check{\zeta}) = \{(\check{\zeta}_1, 0.4e^{i0.6\pi}), (\check{\zeta}_2, 0.6e^{i0.2\pi}), (\check{\zeta}_3, 0.5e^{i0.7\pi})\}$ be a $\mathcal{CFS}$ of U. Then $\bar{v}_{?\tilde{A}}(\check{\zeta}) = \{(\check{\zeta}_1, 0.4e^{i0.5\pi}), (\check{\zeta}_2, 0.5e^{i0.2\pi}), (\check{\zeta}_3, 0.5e^{i0.5\pi})\}$ be a $\mathcal{CFS}$ of U*

Theorem 4. *Let $\tilde{A}$ be a $\mathcal{CFSA}$ of U. Then $?\tilde{A}$ is a $\mathcal{CFSA}$ of U.*

Proof. For each $\check{\zeta}, \in U$, we have

$$\begin{aligned}
\bar{v}_{?\tilde{A}}(0) &= \left(\frac{1}{2} \wedge \gamma_{\tilde{A}}(0)\right) e^{i\left(\frac{1}{2} \wedge \vartheta_{\tilde{A}}(0)\right)} \\
&\geq \left(\frac{1}{2} \wedge \gamma_{\tilde{A}}(\check{\zeta})\right) e^{i\left(\frac{1}{2} \wedge \vartheta_{\tilde{A}}(\check{\zeta})\right)} \\
&= \bar{v}_{?\tilde{A}}(\check{\zeta}).
\end{aligned}$$

Let $\check{\zeta}, \check{\varrho} \in U$. Then

$$\begin{aligned}
\bar{v}_{?\tilde{A}}(\check{\zeta} \diamond \check{\varrho}) &= \frac{1}{2} \wedge \gamma_{\tilde{A}}(\check{\zeta} \diamond \check{\varrho})\, e^{i\left(\vartheta_{\tilde{A}}(\check{\zeta} \diamond \check{\varrho})\right)} \\
&\geq \frac{1}{2} \wedge \left(\gamma_{\tilde{A}}(\check{\zeta}) \wedge \gamma_{\tilde{A}}(\check{\varrho})\right) e^{i\left(\vartheta_{\tilde{A}}(\check{\zeta}) \wedge \vartheta_{\tilde{A}}(\check{\varrho})\right)} \\
&= \left(\frac{1}{2} \wedge \gamma_{\tilde{A}}(\check{\zeta})\, e^{i\vartheta_{\tilde{A}}(\check{\zeta})}\right) \wedge \left(\frac{1}{2} \wedge \gamma_{\tilde{A}}(\check{\varrho})\, e^{i\vartheta_{\tilde{A}}(\check{\varrho})}\right) \\
&= \bar{v}_{?\tilde{A}}(\check{\zeta}) \wedge \bar{v}_{?\tilde{A}}(\check{\varrho}).
\end{aligned}$$

Therefore, $?\tilde{A}$ is a $\mathcal{CFSA}$ of U. □

4. Complex Fuzzy Ideals of BCK/BCI-Algebras (CFIs)

In this section, we will review basic ideas about the $\mathcal{CFS}s$, and $\mathcal{CFI}s$ over the universal set $U \neq \emptyset$.

Definition 9. *A $\mathcal{CFS}$ $\tilde{A} = (\check{\zeta}, \bar{v}_{\tilde{A}}(\check{\zeta}))$ is considered as $\mathcal{CFI}$ of U if $\check{\zeta}, \check{\varrho} \in U$, the following hold:*

1. $\bar{v}_{\tilde{A}}(\check{\zeta}) \geq \bar{v}_{\tilde{A}}(\check{\zeta} \diamond \check{\varrho}) \wedge \bar{v}_{\tilde{A}}(\check{\varrho})$.

Example 7. *Take a BCK-algebra $U = \{0, \check{\zeta}, \check{\varrho}, \check{\kappa}, \check{\varkappa}\}$ with Table 4.*

Now define a $\mathcal{CFS}$ $\tilde{A}$ on U as

$$\tilde{A} = \{(0, 0.67e^{i0.5\pi}), (\check{\zeta}, 0.34e^{i0.43\pi}), (\check{\varrho}, 0.67e^{i0.0.5\pi}), (\check{\kappa}, 0.34e^{i0.43\pi}), (\check{\varkappa}, 0.34e^{i0.43\pi})\}.$$

It is easy to show that $\tilde{A}$ is a $\mathcal{CFI}$ of U.

Table 4. Cayley table representing the binary operation denoted by "$\diamond$".

$\diamond$	**0**	$\check{\zeta}$	$\check{\varrho}$	$\check{\kappa}$	$\check{\varkappa}$
0	0	0	0	0	0
$\check{\zeta}$	$\check{\zeta}$	0	$\check{\zeta}$	0	0
$\check{\varrho}$	$\check{\varrho}$	$\check{\varrho}$	0	0	0
$\check{\kappa}$	$\check{\kappa}$	$\check{\kappa}$	$\check{\kappa}$	0	0
$\check{\varkappa}$	$\check{\varkappa}$	$\check{\kappa}$	$\check{\varkappa}$	$\check{\zeta}$	0

Property 2. *Every $\mathcal{CFI}$ of U is order-preserving.*

Proof. Let $\tilde{A}$ be a $\mathcal{CFSA}$ of U and let $\check{\zeta}, \check{\varrho} \in U$ be such that $\check{\zeta} \leq \check{\varrho}$. Then

$$\begin{aligned}
\bar{v}_{\tilde{A}}(\check{\zeta}) &= \gamma_{\tilde{A}}(\check{\zeta}) e^{i\vartheta_{\tilde{A}}(\check{\zeta})} \\
&\geq \gamma_{\tilde{A}}(\check{\zeta} \diamond \check{\varrho}) e^{i\vartheta_{\tilde{A}}(\check{\zeta} \diamond \check{\varrho})} \wedge \gamma_{\tilde{A}}(\check{\varrho}) e^{i\vartheta_{\tilde{A}}(\check{\varrho})} \\
&= (\gamma_{\tilde{A}}(\check{\zeta} \diamond \check{\varrho}) \wedge \gamma_{\tilde{A}}(\check{\varrho})) e^{i(\vartheta_{\tilde{A}}(\check{\zeta} \diamond \check{\varrho}) \wedge \vartheta_{\tilde{A}}(\check{\varrho}))} \\
&= (\gamma_{\tilde{A}}(0) \wedge \gamma_{\tilde{A}}(\check{\varrho})) e^{i(\vartheta_{\tilde{A}}(0) \wedge \vartheta_{\tilde{A}}(\check{\varrho}))} \\
&= \gamma_{\tilde{A}}(\check{\varrho}) e^{i\vartheta_{\tilde{A}}(\check{\varrho})} \\
&\geq \bar{v}_{\tilde{A}}(\check{\varrho}).
\end{aligned}$$

□

Theorem 5. *Every $\mathcal{CFI}$ of U is an $\mathcal{CFSA}$ of U.*

Proof. Since $\check{\zeta} \diamond \check{\varrho} \leq \check{\zeta}$, it follows from Property 2 that $\bar{v}_{\tilde{A}}(\check{\zeta} \diamond \check{\varrho}) \geq \bar{v}_{\tilde{A}}(\check{\zeta})$. Hence, by Definition 9,

$$\begin{aligned}
\bar{v}_{\tilde{A}}(\check{\zeta} \diamond \check{\varrho}) &\geq \gamma_{\tilde{A}}(\check{\zeta}) e^{i\vartheta_{\tilde{A}}(\check{\zeta})} \\
&= (\gamma_{\tilde{A}}(\check{\zeta} \diamond \check{\varrho}) e^{i\vartheta_{\tilde{A}}(\check{\zeta} \diamond \check{\varrho})}) \wedge (\gamma_{\tilde{A}}(\check{\varrho}) e^{i\vartheta_{\tilde{A}}(\check{\varrho})}) \\
&= (\gamma_{\tilde{A}}(\check{\zeta} \diamond \check{\varrho}) \wedge \gamma_{\tilde{A}}(\check{\varrho})) e^{i(\vartheta_{\tilde{A}}(\check{\zeta} \diamond \check{\varrho}) \wedge \vartheta_{\tilde{A}}(\check{\varrho}))} \\
&= (\gamma_{\tilde{A}}(\check{\zeta}) \wedge \gamma_{\tilde{A}}(\check{\varrho})) e^{i(\vartheta_{\tilde{A}}(\check{\zeta}) \wedge \vartheta_{\tilde{A}}(\check{\varrho}))} \\
&\geq \bar{v}_{\tilde{A}}(\check{\zeta}) \wedge \bar{v}_{\tilde{A}}(\check{\varrho}),
\end{aligned}$$

and so $\tilde{\text{A}}$ is a $\mathcal{CFSA}$ of U. $\square$

Theorem 6. *Let $\tilde{\text{A}}$ be a $\mathcal{CFI}$ of U. If the inequality $\breve{\zeta} \diamond \breve{\varrho} \leq \breve{\kappa}$ holds in U, then $\bar{\upsilon}_{\tilde{\text{A}}}(\breve{\zeta}) \geq \bar{\upsilon}_{\tilde{\text{A}}}(\breve{\varrho}) \wedge \bar{\upsilon}_{\tilde{\text{A}}}(\breve{\kappa})$.*

Proof. Let $\tilde{\text{A}}$ be a $\mathcal{CFI}$ of U and assume that $\breve{\zeta} \diamond \breve{\varrho} \leq \breve{\kappa}$ holds in U. Then

$$\begin{aligned}
\bar{\upsilon}_{\tilde{\text{A}}}(\breve{\zeta} \diamond \breve{\varrho}) &= \gamma_{\tilde{\text{A}}}(\breve{\zeta} \diamond \breve{\varrho}) e^{i\vartheta_{\tilde{\text{A}}}(\breve{\zeta} \diamond \breve{\varrho})} \\
&\geq (\gamma_{\tilde{\text{A}}}((\breve{\zeta} \diamond \breve{\varrho}) \diamond \breve{\kappa}) \wedge \gamma_{\tilde{\text{A}}}(\breve{\kappa}))\, e^{i\left(\vartheta_{\tilde{\text{A}}}((\breve{\zeta} \diamond \breve{\varrho}) \diamond \breve{\kappa}) \wedge \vartheta_{\tilde{\text{A}}}(\breve{\kappa})\right)} \\
&= (\gamma_{\tilde{\text{A}}}(0) \wedge \gamma_{\tilde{\text{A}}}(\breve{\kappa}))\, e^{i\left(\vartheta_{\tilde{\text{A}}}(0) \wedge \vartheta_{\tilde{\text{A}}}(\breve{\kappa})\right)} \\
&= \gamma_{\tilde{\text{A}}}(\breve{\kappa})\, e^{i\vartheta_{\tilde{\text{A}}}(\breve{\kappa})} \\
&\geq \bar{\upsilon}_{\tilde{\text{A}}}(\breve{\kappa}).
\end{aligned}$$

It follows that, $\bar{\upsilon}_{\tilde{\text{A}}}(\breve{\zeta}) \geq \bar{\upsilon}_{\tilde{\text{A}}}(\breve{\kappa}) \wedge \bar{\upsilon}_{\tilde{\text{A}}}(\breve{\varrho})$. $\square$

Definition 10. *Let $\tilde{\text{A}}$ be a $\mathcal{CFS}$ of U. Then, the complement $\tilde{\text{A}}$ is defined as*

$$C(\bar{\upsilon}_{\tilde{\text{A}}}(\breve{\zeta})) = (1 - \gamma_{\tilde{\text{A}}}(\breve{\zeta})) e^{i(2\pi - \vartheta_{\tilde{\text{A}}}(\breve{\zeta}))}.$$

Example 8. *Let*
$\bar{\upsilon}_{\tilde{\text{A}}}(\breve{\zeta})$

$$= \{(\breve{\varsigma}_1, 0.3e^{i0.4\pi}), (\breve{\varsigma}_2, 0.6e^{i0.2\pi}), (\breve{\varsigma}_3, 0.8e^{i0.1\pi}), (\breve{\varsigma}_4, 0.2e^{i0.3\pi}), (\breve{\varsigma}_5, 0.5e^{i\pi}), (\breve{\varsigma}_6, 0.9e^{i0.1\pi})\}$$

be a $\mathcal{CFS}$. Then,
$C(\bar{\upsilon}_{\tilde{\text{A}}}(\breve{\zeta}))$

$$= \{(\breve{\varsigma}_1, 0.7e^{i1.6\pi}), (\breve{\varsigma}_2, 0.4e^{i1.8\pi}), (\breve{\varsigma}_3, 0.2e^{i1.9\pi}), (\breve{\varsigma}_4, 0.8e^{i1.7\pi}), (\breve{\varsigma}_5, 0.5e^{i\pi}), (\breve{\varsigma}_6, 0.1e^{i1.9\pi})\}.$$

Property 3. *A $\mathcal{CFS}$ of U is a $\mathcal{CFI}$ of U iff $C(\bar{\upsilon}_{\tilde{\text{A}}})$ is a $\mathcal{CFI}$ of U.*

Proof. Let $\tilde{\text{A}}$ be a $\mathcal{CFI}$ of U and let $\breve{\zeta}, \breve{\varrho} \in U$. Then

$$\begin{aligned}
C(\bar{\upsilon}_{\tilde{\text{A}}}(0)) &= 1 - \bar{\upsilon}_{\tilde{\text{A}}}(0) \\
&= (1 - \gamma_{\tilde{\text{A}}}(0)) e^{i(2\pi - \vartheta_{\tilde{\text{A}}}(0))} \\
&\geq (1 - \gamma_{\tilde{\text{A}}}(\breve{\zeta})) e^{i(2\pi - \vartheta_{\tilde{\text{A}}}(\breve{\zeta}))} \\
&= C(\bar{\upsilon}_{\tilde{\text{A}}}(\breve{\zeta}))
\end{aligned}$$

and

$$\begin{aligned}
C(\bar{\upsilon}_{\tilde{\text{A}}}(\breve{\zeta})) &= 1 - \bar{\upsilon}_{\tilde{\text{A}}}(\breve{\zeta}) \\
&\geq 1 - (\gamma_{\tilde{\text{A}}}(\breve{\zeta} \diamond \breve{\varrho}) e^{i(2\pi - \vartheta_{\tilde{\text{A}}}(\breve{\zeta} \diamond \breve{\varrho}))} \wedge \gamma_{\tilde{\text{A}}}(\breve{\varrho}) e^{i(2\pi - \vartheta_{\tilde{\text{A}}}(\breve{\varrho}))}) \\
&= (1 - \gamma_{\tilde{\text{A}}}(\breve{\zeta} \diamond \breve{\varrho})) e^{i(2\pi - \vartheta_{\tilde{\text{A}}}(\breve{\zeta} \diamond \breve{\varrho}))} \wedge (1 - \gamma_{\tilde{\text{A}}}(\breve{\varrho})) e^{i(2\pi - \vartheta_{\tilde{\text{A}}}(\breve{\varrho}))} \\
&\geq C(\bar{\upsilon}_{\tilde{\text{A}}}(\breve{\zeta} \diamond \breve{\varrho})) \wedge C(\bar{\upsilon}_{\tilde{\text{A}}}(\breve{\varrho})).
\end{aligned}$$

Thus $C(\bar{\upsilon}_{\tilde{A}})$ is a $\mathcal{CFI}$ of U. The converse of the theorem can be proven in a similar way. □

Definition 11. *Let $\tilde{A}_1$ and $\tilde{A}_2$ be two $\mathcal{CFS}$s of U. Then, the union $\tilde{A}_1 \cup \tilde{A}_2$ is defined as*

$$\bar{\upsilon}_{(\tilde{A}_1 \cup \tilde{A}_2)}(\check{\zeta}) = (\gamma_{\tilde{A}_1}(\check{\zeta}) \vee \gamma_{\tilde{A}_2}(\check{\zeta}))e^{i(\vartheta_{\tilde{A}_1}(\check{\zeta}) \vee \vartheta_{\tilde{A}_2}(\check{\zeta}))}$$

Example 9. *Let*

$\bar{\upsilon}_{\tilde{A}_1}(\check{\zeta}) = \{(\check{\zeta}_1, 0.6e^{i0.5\pi}), (\check{\zeta}_2, 1e^{i0.5\pi}), (\check{\zeta}_3, 0.8e^{i2\pi}), (\check{\zeta}_4, 0.9e^{i0.4\pi}), (\check{\zeta}_5, 0.7e^{i\pi}), (\check{\zeta}_6, 0.5e^{i0.4\pi})\}$

and

$\bar{\upsilon}_{\tilde{A}_2}(\check{\zeta})$

$= \{(\check{\zeta}_1, 0.2e^{i\pi}), (\check{\zeta}_2, 0.1e^{i0.8\pi}), (\check{\zeta}_3, 0.8e^{i0.8\pi}), (\check{\zeta}_4, 0.2e^{i0.9\pi}), (\check{\zeta}_5, 0.9e^{i0.9\pi}), (\check{\zeta}_6, 0.3e^{i2\pi})\}$

be a $\mathcal{CFS}$s. Then,

$\bar{\upsilon}_{(\tilde{A}_1 \cup \tilde{A}_2)}(\check{\zeta}) = \{(\check{\zeta}_1, 0.6e^{i\pi}), (\check{\zeta}_2, 1e^{i0.8\pi}), (\check{\zeta}_3, 0.8e^{i2\pi}), (\check{\zeta}_4, 0.9e^{i\pi}), (\check{\zeta}_5, 0.9e^{i\pi}), (\check{\zeta}_6, 0.5e^{i2\pi})\}.$

Property 4. *Let $\tilde{A}_1$ and $\tilde{A}_2$ be two $\mathcal{CFI}$s of U. Then $\tilde{A}_1 \cup \tilde{A}_2$ is a $\mathcal{CFI}$ of U.*

Proof. Let $\tilde{A}_1$ and $\tilde{A}_2$ be two $\mathcal{CFI}s$ of U and let $\check{\zeta}, \check{\varrho} \in U$. Then

$$\begin{aligned}\bar{\upsilon}_{(\tilde{A}_1 \cup \tilde{A}_2)}(0) &= (\gamma_{\tilde{A}_1}(0) \vee \gamma_{\tilde{A}_2}(0))e^{i(\vartheta_{\tilde{A}_1}(0) \vee \vartheta_{\tilde{A}_2}(0))} \\ &\geq (\gamma_{\tilde{A}_1}(\check{\zeta}) \vee \gamma_{\tilde{A}_2}(\check{\zeta}))e^{i(\vartheta_{\tilde{A}_1}(\check{\zeta}) \vee \vartheta_{\tilde{A}_2}(\check{\zeta}))} \\ &= \bar{\upsilon}_{(\tilde{A}_1 \cup \tilde{A}_2)}(\check{\zeta})\end{aligned}$$

and

$$\begin{aligned}\bar{\upsilon}_{(\tilde{A}_1 \cup \tilde{A}_2)}(\check{\zeta}) &= (\gamma_{\tilde{A}_1}(\check{\zeta}) \vee \gamma_{\tilde{A}_2}(\check{\zeta}))e^{i(\vartheta_{\tilde{A}_1}(\check{\zeta}) \vee \vartheta_{\tilde{A}_2}(\check{\zeta}))} \\ &\geq ((\gamma_{\tilde{A}_1}(\check{\zeta} \diamond \check{\varrho}) \wedge \gamma_{\tilde{A}_1}(\check{\varrho})) \vee (\gamma_{\tilde{A}_2}(\check{\zeta} \diamond \check{\varrho}) \wedge \gamma_{\tilde{A}_2}(\check{\varrho}))) \\ &\quad e^{i((\vartheta_{\tilde{A}_1}(\check{\zeta} \diamond \check{\varrho}) \wedge \vartheta_{\tilde{A}_1}(\check{\varrho})) \vee (\vartheta_{\tilde{A}_2}(\check{\zeta} \diamond \check{\varrho}) \wedge \vartheta_{\tilde{A}_2}(\check{\varrho})))}\end{aligned}$$

If one is contained in the other, then

$$\begin{aligned}&\geq ((\gamma_{\tilde{A}_1}(\check{\zeta} \diamond \check{\varrho}) \vee \gamma_{\tilde{A}_2}(\check{\zeta} \diamond \check{\varrho})) \wedge (\gamma_{\tilde{A}_1}(\check{\varrho}) \vee \gamma_{\tilde{A}_2}(\check{\varrho}))) \\ &\quad e^{i((\vartheta_{\tilde{A}_1}(\check{\zeta} \diamond \check{\varrho}) \vee \vartheta_{\tilde{A}_2}(\check{\zeta} \diamond \check{\varrho})) \wedge (\vartheta_{\tilde{A}_1}(\check{\varrho}) \vee \vartheta_{\tilde{A}_2}(\check{\varrho})))} \\ &= (\gamma_{\tilde{A}_1}(\check{\zeta} \diamond \check{\varrho}) \vee \gamma_{\tilde{A}_2}(\check{\zeta} \diamond \check{\varrho}))e^{i(\vartheta_{\tilde{A}_1}(\check{\zeta} \diamond \check{\varrho}) \vee \vartheta_{\tilde{A}_2}(\check{\zeta} \diamond \check{\varrho}))} \wedge \\ &\quad (\gamma_{\tilde{A}_1}(\check{\varrho}) \vee \gamma_{\tilde{A}_2}(\check{\varrho}))e^{i(\vartheta_{\tilde{A}_1}(\check{\varrho}) \vee \vartheta_{\tilde{A}_2}(\check{\varrho}))} \\ &\geq \bar{\upsilon}_{(\tilde{A}_1 \cup \tilde{A}_2)}(\check{\zeta} \diamond \check{\varrho}) \wedge \bar{\upsilon}_{(\tilde{A}_1 \cup \tilde{A}_2)}(\check{\varrho}).\end{aligned}$$

Therefore, $\tilde{A}_1 \cup \tilde{A}_2$ is a $\mathcal{CFI}$ of U. □

Example 10. *Take a BCK-algebra $U = \{0, \check{\zeta}, \check{\varrho}, \check{\kappa}, \check{\varkappa}\}$ with Table 5.*

Now define $\mathcal{CFS}$ $\tilde{A}_1$ on U as

U	0	$\check{\zeta}$	$\check{\varrho}$	$\check{\kappa}$	$\check{\varkappa}$
$\bar{\upsilon}_{\tilde{A}_1}(\check{\zeta})$	$0.9e^{i0.7\pi}$	$0.7e^{i0.5\pi}$	$0.5e^{i0.3\pi}$	$0.3e^{i0.1\pi}$	$0.3e^{i0.1\pi}$

It is easy to show that $\tilde{A}_1$ is a $\mathcal{CFI}$ of U.
Now define $\mathcal{CFS}$ $\tilde{A}_2$ on U as

U	0	$\check{\zeta}$	$\check{\varrho}$	$\check{\kappa}$	$\check{\varkappa}$
$\bar{v}_{\tilde{A}_2}(\check{\zeta})$	$0.6e^{i0.5\pi}$	$0.4e^{i0.6\pi}$	$0.6e^{i0.5\pi}$	$0.4e^{i0.6\pi}$	$0.4e^{i0.6\pi}$

It is easy to show that $\tilde{A}_2$ is a $\mathcal{CFI}$ of U.
Now define $\mathcal{CFS}$ $\tilde{A}_1 \cup \tilde{A}_2$ on U as

U	0	$\check{\zeta}$	$\check{\varrho}$	$\check{\kappa}$	$\check{\varkappa}$
$\bar{v}_{(\tilde{A}_1 \cup \tilde{A}_2)}(\check{\zeta})$	$0.9e^{i0.7\pi}$	$0.7e^{i0.6\pi}$	$0.6e^{i0.5\pi}$	$0.4e^{i0.6\pi}$	$0.4e^{i0.6\pi}$

It is easy to show that $\tilde{A}_1 \cup \tilde{A}_2$ is a $\mathcal{CFI}$ of U.

Table 5. Cayley's table representing the binary operation denoted by "$\diamond$".

$\diamond$	**0**	$\check{\zeta}$	$\check{\varrho}$	$\check{\kappa}$	$\check{\varkappa}$
0	0	0	0	0	0
$\check{\zeta}$	$\check{\zeta}$	0	$\check{\zeta}$	0	0
$\check{\varrho}$	$\check{\varrho}$	$\check{\varrho}$	0	0	0
$\check{\kappa}$	$\check{\kappa}$	$\check{\kappa}$	$\check{\kappa}$	0	0
$\check{\varkappa}$	$\check{\varkappa}$	$\check{\varkappa}$	$\check{\varkappa}$	$\check{\kappa}$	0

Definition 12. *Let $\tilde{A}_1$ and $\tilde{A}_2$ be two $\mathcal{CFS}$s of U. Then, the intersection $\tilde{A}_1 \cap \tilde{A}_2$ is defined as*

$$\bar{v}_{(\tilde{A}_1 \cap \tilde{A}_2)}(\check{\zeta}) = (\gamma_{\tilde{A}_1}(\check{\zeta}) \wedge \gamma_{\tilde{A}_2}(\check{\zeta}))e^{i(\vartheta_{\tilde{A}_1}(\check{\zeta}) \wedge \vartheta_{\tilde{A}_2}(\check{\zeta}))}$$

Example 11. *Let*

$$\bar{v}_{\tilde{A}_1}(\check{\zeta}) = \{(\check{\zeta}_1, 0.6e^{i0.5\pi}), (\check{\zeta}_2, 1e^{i0.5\pi}), (\check{\zeta}_3, 0.8e^{i2\pi}), (\check{\zeta}_4, 0.5e^{i0.4\pi}), (\check{\zeta}_5, 0.9e^{i0.4\pi}), (\check{\zeta}_6, 0.7e^{i\pi})\}$$

and

$$\bar{v}_{\tilde{A}_2}(\check{\zeta}) = \{(\check{\zeta}_1, 0.2e^{i\pi}), (\check{\zeta}_2, 0.1e^{i0.8\pi}), (\check{\zeta}_3, 0.7e^{i0.7\pi}), (\check{\zeta}_4, 0.1e^{i0.9\pi}), (\check{\zeta}_5, 0.9e^{i0.7\pi}), (\check{\zeta}_6, 0.6e^{i0.6\pi})\}$$

be a $\mathcal{CFS}$s. Then,

$$\bar{v}_{(\tilde{A}_1 \cap \tilde{A}_2)}(\check{\zeta}) = \{(\check{\zeta}_1, 0.2e^{i0.5\pi}), (\check{\zeta}_2, 0.1e^{i0.5\pi}), (\check{\zeta}_3, 0.7e^{i0.7\pi}), (\check{\zeta}_4, 0.1e^{i0.4\pi}), (\check{\zeta}_5, 0.9e^{i0.4\pi}), (\check{\zeta}_6, 0.6e^{0.6i\pi})\}.$$

Property 5. *Let $\tilde{A}_1$ and $\tilde{A}_2$ be two $\mathcal{CFI}$s of U. Then $\tilde{A}_1 \cap \tilde{A}_2$ is a $\mathcal{CFI}$ of U.*

Proof. Let $\tilde{A}_1$ and $\tilde{A}_2$ be two $\mathcal{CFI}s$ of U and let $\check{\zeta}, \check{\varrho} \in U$. Then,

$$\begin{aligned}\bar{v}_{(\tilde{A}_1 \cap \tilde{A}_2)}(0) &= (\gamma_{\tilde{A}_1}(0) \wedge \gamma_{\tilde{A}_2}(0))e^{i(\vartheta_{\tilde{A}_1}(0) \wedge \vartheta_{\tilde{A}_2}(0))} \\ &\geq (\gamma_{\tilde{A}_1}(\check{\zeta}) \wedge \gamma_{\tilde{A}_2}(\check{\zeta}))e^{i(\vartheta_{\tilde{A}_1}(\check{\zeta}) \wedge \vartheta_{\tilde{A}_2}(\check{\zeta}))} \\ &= \bar{v}_{(\tilde{A}_1 \cap \tilde{A}_2)}(\check{\zeta})\end{aligned}$$

and

$$\begin{aligned}
\bar{\upsilon}_{(\tilde{A}_1 \cap \tilde{A}_2)}(\check{\zeta}) &= (\gamma_{\tilde{A}_1}(\check{\zeta}) \wedge \gamma_{\tilde{A}_2}(\check{\zeta}))e^{i(\vartheta_{\tilde{A}_1}(\check{\zeta}) \wedge \vartheta_{\tilde{A}_2}(\check{\zeta}))} \\
&\geq ((\gamma_{\tilde{A}_1}(\check{\zeta} \diamond \check{\varrho}) \wedge \gamma_{\tilde{A}_1}(\check{\varrho})) \wedge (\gamma_{\tilde{A}_2}(\check{\zeta} \diamond \check{\varrho}) \wedge \gamma_{\tilde{A}_2}(\check{\varrho}))) \\
&\quad e^{i((\vartheta_{\tilde{A}_1}(\check{\zeta}\diamond\check{\varrho}) \wedge \vartheta_{\tilde{A}_1}(\check{\varrho})) \wedge (\vartheta_{\tilde{A}_2}(\check{\zeta}\diamond\check{\varrho}) \wedge \vartheta_{\tilde{A}_2}(\check{\varrho}))} \\
&= ((\gamma_{\tilde{A}_1}(\check{\zeta} \diamond \check{\varrho}) \wedge \gamma_{\tilde{A}_2}(\check{\zeta} \diamond \check{\varrho})) \wedge (\gamma_{\tilde{A}_1}(\check{\varrho}) \wedge \gamma_{\tilde{A}_2}(\check{\varrho}))) \\
&\quad e^{i((\vartheta_{\tilde{A}_1}(\check{\zeta}\diamond\check{\varrho}) \wedge \vartheta_{\tilde{A}_2}(\check{\zeta}\diamond\check{\varrho})) \wedge (\vartheta_{\tilde{A}_1}(\check{\varrho}) \wedge \vartheta_{\tilde{A}_2}(\check{\varrho})))} \\
&= (\gamma_{\tilde{A}_1}(\check{\zeta} \diamond \check{\varrho}) \wedge \gamma_{\tilde{A}_2}(\check{\zeta} \diamond \check{\varrho}))e^{i(\vartheta_{\tilde{A}_1}(\check{\zeta}\diamond\check{\varrho}) \wedge \vartheta_{\tilde{A}_2}(\check{\zeta}\diamond\check{\varrho}))} \wedge \\
&\quad (\gamma_{\tilde{A}_1}(\check{\varrho}) \wedge \gamma_{\tilde{A}_2}(\check{\varrho}))e^{i(\vartheta_{\tilde{A}_1}(\check{\varrho}) \wedge \vartheta_{\tilde{A}_2}(\check{\varrho}))} \\
&\geq \bar{\upsilon}_{(\tilde{A}_1 \cap \tilde{A}_2)}(\check{\zeta} \diamond \check{\varrho}) \wedge \bar{\upsilon}_{(\tilde{A}_1 \cap \tilde{A}_2)}(\check{\varrho}).
\end{aligned}$$

Therefore, $\tilde{A}_1 \cap \tilde{A}_2$ is a $\mathcal{CFI}$ of U. □

Example 12. *Take a BCK-algebra $U = \{0, \check{\zeta}, \check{\varrho}, \check{\kappa}, \check{\varkappa}\}$ with Table 6.*
Now define $\mathcal{CFS}$ $\tilde{A}_1$ on U as

U	0	$\check{\zeta}$	$\check{\varrho}$	$\check{\kappa}$	$\check{\varkappa}$
$\bar{\upsilon}_{\tilde{A}_1}(\check{\zeta})$	$0.6e^{i0.8\pi}$	$0.5e^{i0.7\pi}$	$0.4e^{i0.6\pi}$	$0.4e^{i0.6\pi}$	$0.4e^{i0.6\pi}$

It is easy to show that $\tilde{A}_1$ is a $\mathcal{CFI}$ of U.
Now define $\mathcal{CFS}$ $\tilde{A}_2$ on U as

U	0	$\check{\zeta}$	$\check{\varrho}$	$\check{\kappa}$	$\check{\varkappa}$
$\bar{\upsilon}_{\tilde{A}_2}(\check{\zeta})$	$0.8e^{i0.6\pi}$	$0.7e^{i0.5\pi}$	$0.6e^{i0.3\pi}$	$0.6e^{i0.3\pi}$	$0.6e^{i0.3\pi}$

It is easy to show that $\tilde{A}_2$ is a $\mathcal{CFI}$ of U.
Now define $\mathcal{CFS}$ $\tilde{A}_1 \cap \tilde{A}_2$ on U as

U	0	$\check{\zeta}$	$\check{\varrho}$	$\check{\kappa}$	$\check{\varkappa}$
$\bar{\upsilon}_{(\tilde{A}_1 \cap \tilde{A}_2)}(\check{\zeta})$	$0.6e^{i0.6\pi}$	$0.5e^{i0.5\pi}$	$0.4e^{i0.3\pi}$	$0.4e^{i0.3\pi}$	$0.4e^{i0.3\pi}$

It is easy to show that $\tilde{A}_1 \cap \tilde{A}_2$ is a $\mathcal{CFI}$ of U.

Table 6. Cayley's table representing the binary operation denoted by "$\diamond$".

$\diamond$	**0**	$\check{\zeta}$	$\check{\varrho}$	$\check{\kappa}$	$\check{\varkappa}$
0	0	0	0	0	0
$\check{\zeta}$	$\check{\zeta}$	0	$\check{\zeta}$	0	0
$\check{\varrho}$	$\check{\varrho}$	$\check{\varrho}$	0	0	0
$\check{\kappa}$	$\check{\kappa}$	$\check{\kappa}$	$\check{\kappa}$	0	0
$\check{\varkappa}$	$\check{\varkappa}$	$\check{\varkappa}$	$\check{\kappa}$	$\check{\varrho}$	0

Definition 13. *Let $\tilde{A}_1$ and $\tilde{A}_2$ be two $\mathcal{CFS}$s of U. Then, the simple difference $\tilde{A}_1 \setminus \tilde{A}_2$ is defined as*

$$\bar{\upsilon}_{(\tilde{A}_1 \setminus \tilde{A}_2)}(\check{\zeta}) = (\gamma_{\tilde{A}_1}(\check{\zeta}) \wedge \gamma_{\tilde{A}_2}(\check{\zeta}))e^{i(\vartheta_{\tilde{A}_1}(\check{\zeta}) \wedge \vartheta_{\tilde{A}_2}(\check{\zeta}))}$$

Example 13. *Let*

$$\bar{\upsilon}_{\tilde{A}_1}(\check{\zeta})$$
$$= \{(\check{\zeta}_1, 0.8e^{i0.1\pi}), (\check{\zeta}_2, 0.4e^{i0.6\pi}), (\check{\zeta}_3, 1e^{i\pi}), (\check{\zeta}_4, 0.7e^{i0.2\pi}), (\check{\zeta}_5, 0.3e^{i0.5\pi}), (\check{\zeta}_6, 0.9e^{i0.9\pi})\}$$

and

$\bar{\upsilon}_{\tilde{A}_2}(\breve{\zeta})$

$$= \{(\breve{\varsigma}_1, 0.2e^{i\pi}), (\breve{\varsigma}_2, 1e^{i0.8\pi}), (\breve{\varsigma}_3, 0.5e^{i0.2\pi}), (\breve{\varsigma}_4, 0.1e^{i0.9\pi}), (\breve{\varsigma}_5, 0.9e^{i0.7\pi}), (\breve{\varsigma}_6, 0.4e^{i0.1\pi})\}$$

be a $\mathcal{CFS}$s. Then,

$\bar{\upsilon}_{(\tilde{A}_1 \setminus \tilde{A}_2)}(\breve{\zeta}) =$

$$\{(\breve{\varsigma}_1, 0.2e^{i0.1\pi}), (\breve{\varsigma}_2, 0.4e^{i0.6\pi}), (\breve{\varsigma}_3, 0.5e^{i0.2\pi}), (\breve{\varsigma}_4, 0.1e^{i0.2\pi}), (\breve{\varsigma}_5, 0.3e^{i0.5\pi}), (\breve{\varsigma}_6, 0.4e^{i0.1\pi})\}.$$

Property 6. *Let $\tilde{A}_1$ and $\tilde{A}_2$ be two $\mathcal{CFI}$s of U. Then $\tilde{A}_1 \setminus \tilde{A}_2$ is a $\mathcal{CFI}$ of U.*

Proof. Let $\tilde{A}_1$ and $\tilde{A}_2$ be two $\mathcal{CFI}s$ of U and let $\breve{\zeta}, \breve{\varrho} \in U$. Then

$$\begin{aligned}
\bar{\upsilon}_{(\tilde{A}_1 \setminus \tilde{A}_2)}(0) &= (\gamma_{\tilde{A}_1}(0) \wedge \gamma_{\tilde{A}_2}(0)) e^{i(\vartheta_{\tilde{A}_1}(0) \wedge \vartheta_{\tilde{A}_2}(0))} \\
&\geq (\gamma_{\tilde{A}_1}(\breve{\zeta}) \wedge \gamma_{\tilde{A}_2}(\breve{\zeta})) e^{i(\vartheta_{\tilde{A}_1}(\breve{\zeta}) \wedge \vartheta_{\tilde{A}_2}(\breve{\zeta}))} \\
&= \bar{\upsilon}_{(\tilde{A}_1 \setminus \tilde{A}_2)}(\breve{\zeta})
\end{aligned}$$

and

$$\begin{aligned}
\bar{\upsilon}_{(\tilde{A}_1 \setminus \tilde{A}_2)}(\breve{\zeta}) &= (\gamma_{\tilde{A}_1}(\breve{\zeta}) \wedge \gamma_{\tilde{A}_2}(\breve{\zeta})) e^{i(\vartheta_{\tilde{A}_1}(\breve{\zeta}) \wedge \vartheta_{\tilde{A}_2}(\breve{\zeta}))} \\
&\geq ((\gamma_{\tilde{A}_1}(\breve{\zeta} \diamond \breve{\varrho}) \wedge \gamma_{\tilde{A}_1}(\breve{\varrho})) \wedge (\gamma_{\tilde{A}_2}(\breve{\zeta} \diamond \breve{\varrho}) \wedge \gamma_{\tilde{A}_2}(\breve{\varrho}))) \\
&\quad e^{i((\vartheta_{\tilde{A}_1}(\breve{\zeta} \diamond \breve{\varrho}) \wedge \vartheta_{\tilde{A}_1}(\breve{\varrho})) \wedge (\vartheta_{\tilde{A}_2}(\breve{\zeta} \diamond \breve{\varrho}) \wedge \vartheta_{\tilde{A}_2}(\breve{\varrho}))} \\
&= ((\gamma_{\tilde{A}_1}(\breve{\zeta} \diamond \breve{\varrho}) \wedge \gamma_{\tilde{A}_2}(\breve{\zeta} \diamond \breve{\varrho})) \wedge (\gamma_{\tilde{A}_1}(\breve{\varrho}) \wedge \gamma_{\tilde{A}_2}(\breve{\varrho}))) \\
&\quad e^{i((\vartheta_{\tilde{A}_1}(\breve{\zeta} \diamond \breve{\varrho}) \wedge \vartheta_{\tilde{A}_2}(\breve{\zeta} \diamond \breve{\varrho})) \wedge (\vartheta_{\tilde{A}_1}(\breve{\varrho}) \wedge \vartheta_{\tilde{A}_2}(\breve{\varrho})))} \\
&= (\gamma_{\tilde{A}_1}(\breve{\zeta} \diamond \breve{\varrho}) \wedge \gamma_{\tilde{A}_2}(\breve{\zeta} \diamond \breve{\varrho})) e^{i(\vartheta_{\tilde{A}_1}(\breve{\zeta} \diamond \breve{\varrho}) \wedge \vartheta_{\tilde{A}_2}(\breve{\zeta} \diamond \breve{\varrho}))} \wedge \\
&\quad (\gamma_{\tilde{A}_1}(\breve{\varrho}) \wedge \gamma_{\tilde{A}_2}(\breve{\varrho})) e^{i(\vartheta_{\tilde{A}_1}(\breve{\varrho}) \wedge \vartheta_{\tilde{A}_2}(\breve{\varrho}))} \\
&\geq \bar{\upsilon}_{(\tilde{A}_1 \setminus \tilde{A}_2)}(\breve{\zeta} \diamond \breve{\varrho}) \wedge \bar{\upsilon}_{(\tilde{A}_1 \setminus \tilde{A}_2)}(\breve{\varrho}).
\end{aligned}$$

Therefore, $\tilde{A}_1 \setminus \tilde{A}_2$ is a $\mathcal{CFI}$ of U. □

Example 14. *Take a BCK-algebra $U = \{0, \breve{\zeta}, \breve{\varrho}, \breve{\kappa}, \breve{\varkappa}\}$ with Table 7.*

Now define $\mathcal{CFS}$ $\tilde{A}_1$ on U as

U	0	$\breve{\zeta}$	$\breve{\varrho}$	$\breve{\kappa}$
$\bar{\upsilon}_{\tilde{A}_1}(\breve{\zeta})$	$0.8e^{i0.7\pi}$	$0.6e^{i0.5\pi}$	$0.2e^{i0.3\pi}$	$0.2e^{i0.3\pi}$

It is easy to show that $\tilde{A}_1$ is a $\mathcal{CFI}$ of U.

Now define $\mathcal{CFS}$ $\tilde{A}_2$ on U as

U	0	$\breve{\zeta}$	$\breve{\varrho}$	$\breve{\kappa}$
$\bar{\upsilon}_{\tilde{A}_2}(\breve{\zeta})$	$0.5e^{i0.8\pi}$	$0.5e^{i0.8\pi}$	$0.3e^{i0.6\pi}$	$0.3e^{i0.6\pi}$

It is easy to show that $\tilde{A}_2$ is a $\mathcal{CFI}$ of U.

Now define $\mathcal{CFS}$ $\tilde{A}_1 \setminus \tilde{A}_2$ on U as

U	0	$\breve{\zeta}$	$\breve{\varrho}$	$\breve{\kappa}$
$\bar{\upsilon}_{(\tilde{A}_1 \setminus \tilde{A}_2)}(\breve{\zeta})$	$0.5e^{i0.7\pi}$	$0.5e^{i0.5\pi}$	$0.2e^{i0.3\pi}$	$0.2e^{i0.3\pi}$

It is easy to show that $\tilde{A}_1 \setminus \tilde{A}_2$ *is a* $\mathcal{CFI}$ *of* U.

Table 7. Cayley's table representing the binary operation denoted by "$\diamond$".

$\diamond$	**0**	$\check{\zeta}$	$\check{\varrho}$	$\check{\kappa}$
0	0	0	0	0
$\check{\zeta}$	$\check{\zeta}$	0	0	$\check{\zeta}$
$\check{\varrho}$	$\check{\varrho}$	$\check{\varrho}$	0	0
$\check{\kappa}$	$\check{\kappa}$	$\check{\kappa}$	$\check{\kappa}$	0

Definition 14. *Let* $\tilde{A}_1$ *and* $\tilde{A}_2$ *be two* $\mathcal{CFS}$*s of* U*. Then, the bounded difference* $\tilde{A}_1 \ominus \tilde{A}_2$ *is defined as*

$$\bar{v}_{(\tilde{A}_1 \ominus \tilde{A}_2)}(\check{\zeta}) = (0 \vee (\gamma_{\tilde{A}_1}(\check{\zeta}) - \gamma_{\tilde{A}_2}(\check{\zeta})))e^{i(\vartheta_{\tilde{A}_1}(\check{\zeta}) \vee \vartheta_{\tilde{A}_2}(\check{\zeta}))}$$

Example 15. *Let*

$\bar{v}_{\tilde{A}_1}(\check{\zeta}) = \{(\check{\zeta}_1, 0.6e^{i0.5\pi}), (\check{\zeta}_2, 1e^{i0.5\pi}), (\check{\zeta}_3, 0.8e^{i2\pi}), (\check{\zeta}_4, 0.5e^{i0.4\pi}), (\check{\zeta}_5, 0.9e^{i0.4\pi}), (\check{\zeta}_6, 0.7e^{i\pi})\}$

and

$$\bar{v}_{\tilde{A}_2}(\check{\zeta}) = \{(\check{\zeta}_1, 0.2e^{i\pi}), (\check{\zeta}_2, 0.1e^{i0.8\pi}), (\check{\zeta}_3, 0.8e^{i0.8\pi}), (\check{\zeta}_4, 0.1e^{i0.9\pi}), (\check{\zeta}_5, 0e^{i0.7\pi}), (\check{\zeta}_6, 0.7e^{i0.7\pi})\}$$

be a $\mathcal{CFS}$*s. Then*

$$\bar{v}_{(\tilde{A}_1 \ominus \tilde{A}_2)}(\check{\zeta}) = \{(\check{\zeta}_1, 0.4e^{i\pi}), (\check{\zeta}_2, 0.9e^{i0.8\pi}), (\check{\zeta}_3, 0e^{i2\pi}), (\check{\zeta}_4, 0.4e^{i0.9\pi}), (\check{\zeta}_5, 0.9e^{i0.7\pi}), (\check{\zeta}_6, 0e^{i\pi})\}.$$

Property 7. *Let* $\tilde{A}_1$ *and* $\tilde{A}_2$ *be two* $\mathcal{CFI}$*s of* U*. Then* $\tilde{A}_1 \ominus \tilde{A}_2$ *is a* $\mathcal{CFI}$ *of* U.

Proof. Let $\tilde{A}_1$ and $\tilde{A}_2$ be two $\mathcal{CFI}s$ of U and let $\check{\zeta}, \check{\varrho} \in U$. Then

$$\begin{aligned}
\bar{v}_{(\tilde{A}_1 \ominus \tilde{A}_2)}(0) &= (0 \vee (\gamma_{\tilde{A}_1}(0) - \gamma_{\tilde{A}_2}(0)))e^{i(\vartheta_{\tilde{A}_1}(0) \vee \vartheta_{\tilde{A}_2}(0))} \\
&\geq (0 \vee (\gamma_{\tilde{A}_1}(\check{\zeta}) - \gamma_{\tilde{A}_2}(\check{\zeta})))e^{i(\vartheta_{\tilde{A}_1}(\check{\zeta}) \vee \vartheta_{\tilde{A}_2}(\check{\zeta}))} \\
&= \bar{v}_{(\tilde{A}_1 \ominus \tilde{A}_2)}(\check{\zeta})
\end{aligned}$$

and

$$\begin{aligned}
\bar{v}_{(\tilde{A}_1 \ominus \tilde{A}_2)}(\check{\zeta}) &= (0 \vee (\gamma_{\tilde{A}_1}(\check{\zeta}) - \gamma_{\tilde{A}_2}(\check{\zeta})))e^{i(\vartheta_{\tilde{A}_1}(\check{\zeta}) \vee \vartheta_{\tilde{A}_2}(\check{\zeta}))} \\
&\geq (0 \vee ((\gamma_{\tilde{A}_1}(\check{\zeta} \diamond \check{\varrho}) - \gamma_{\tilde{A}_2}(\check{\zeta} \diamond \check{\varrho})) \wedge (\gamma_{\tilde{A}_1}(\check{\varrho}) - \gamma_{\tilde{A}_2}(\check{\varrho})))) \\
&\quad e^{i((\vartheta_{\tilde{A}_1}(\check{\zeta} \diamond \check{\varrho}) \wedge \vartheta_{\tilde{A}_1}(\check{\varrho})) \vee (\vartheta_{\tilde{A}_2}(\check{\zeta} \diamond \check{\varrho}) \wedge \vartheta_{\tilde{A}_2}(\check{\varrho}))} \\
&= ((0 \vee (\gamma_{\tilde{A}_1}(\check{\zeta} \diamond \check{\varrho}) - \gamma_{\tilde{A}_2}(\check{\zeta} \diamond \check{\varrho}))) \wedge (0 \vee (\gamma_{\tilde{A}_1}(\check{\varrho}) - \gamma_{\tilde{A}_2}(\check{\varrho})))) \\
&\quad e^{i((\vartheta_{\tilde{A}_1}(\check{\zeta} \diamond \check{\varrho}) \wedge \vartheta_{\tilde{A}_2}(\check{\zeta} \diamond \check{\varrho})) \vee (\vartheta_{\tilde{A}_1}(\check{\varrho}) \wedge \vartheta_{\tilde{A}_2}(\check{\varrho})))} \\
&= (0 \vee (\gamma_{\tilde{A}_1}(\check{\zeta} \diamond \check{\varrho}) - \gamma_{\tilde{A}_2}(\check{\zeta} \diamond \check{\varrho})))e^{i(\vartheta_{\tilde{A}_1}(\check{\zeta} \diamond \check{\varrho}) \vee \vartheta_{\tilde{A}_2}(\check{\zeta} \diamond \check{\varrho}))} \wedge \\
&\quad (0 \vee (\gamma_{\tilde{A}_1}(\check{\varrho}) \ominus \gamma_{\tilde{A}_2}(\check{\varrho})))e^{i(\vartheta_{\tilde{A}_1}(\check{\varrho}) \vee \vartheta_{\tilde{A}_2}(\check{\varrho}))} \\
&\geq \bar{v}_{(\tilde{A}_1 \ominus \tilde{A}_2)}(\check{\zeta} \diamond \check{\varrho}) \wedge \bar{v}_{(\tilde{A}_1 \ominus \tilde{A}_2)}(\check{\varrho}).
\end{aligned}$$

Therefore, $\tilde{A}_1 \ominus \tilde{A}_2$ is a $\mathcal{CFI}$ of U. □

Example 16. *Take a BCK-algebra $U = \{0, \check{\varsigma}, \check{\varrho}, \check{\kappa}, \check{\varkappa}\}$ with Table 8.*
Now define $\mathcal{CFS}$ $\tilde{A}_1$ on U as

U	0	$\check{\varsigma}$	$\check{\varrho}$	$\check{\kappa}$	$\check{\varkappa}$
$\bar{\upsilon}_{\tilde{A}_1}(\check{\varsigma})$	$0.7e^{i2\pi}$	$0.6e^{i1.5\pi}$	$0.3e^{i0.5\pi}$	$0.1e^{i0\pi}$	$0.2e^{i\pi}$

It is easy to show that $\tilde{A}_1$ is a $\mathcal{CFI}$ of U.
Now define $\mathcal{CFS}$ $\tilde{A}_2$ on U as

U	0	$\check{\varsigma}$	$\check{\varrho}$	$\check{\kappa}$	$\check{\varkappa}$
$\bar{\upsilon}_{\tilde{A}_2}(\check{\varsigma})$	$0.6e^{i\pi}$	$0.4e^{i0.8\pi}$	$0.2e^{i0.4\pi}$	$0e^{i0.2\pi}$	$0.2e^{i0.6\pi}$

It is easy to show that $\tilde{A}_2$ is a $\mathcal{CFI}$ of U.
Now define $\mathcal{CFS}$ $\tilde{A}_1 \ominus \tilde{A}_2$ on U as

U	0	$\check{\varsigma}$	$\check{\varrho}$	$\check{\kappa}$	$\check{\varkappa}$
$\bar{\upsilon}_{(\tilde{A}_1 \ominus \tilde{A}_2)}(\check{\varsigma})$	$0.1e^{i\pi}$	$0.1e^{i1.5\pi}$	$0.1e^{i0.5\pi}$	$0.1e^{i0.2\pi}$	$0e^{i\pi}$

It is easy to show that $\tilde{A}_1 \ominus \tilde{A}_2$ is a $\mathcal{CFI}$ of U.

Table 8. Cayley's table representing the binary operation denoted by "$\diamond$".

$\diamond$	**0**	$\check{\varsigma}$	$\check{\varrho}$	$\check{\kappa}$	$\check{\varkappa}$
0	0	0	0	0	0
$\check{\varsigma}$	$\check{\varsigma}$	0	$\check{\varsigma}$	0	0
$\check{\varrho}$	$\check{\varrho}$	$\check{\varrho}$	0	$\check{\varrho}$	0
$\check{\kappa}$	$\check{\kappa}$	$\check{\kappa}$	$\check{\kappa}$	0	$\check{\kappa}$
$\check{\varkappa}$	$\check{\varkappa}$	$\check{\varkappa}$	$\check{\varkappa}$	$\check{\varkappa}$	0

5. Comparison Analysis of the Proposed Approach

We provide a theoretical description of the comparison analysis of our proposed approach using the complex fuzzy sets method proposed in [16,17].

1. Representation and Membership:
   - The complex fuzzy sets and ideals represent the degree of membership in complex numbers.
2. Algebraic Structure:
   - Complex fuzzy sets are more general constructs that can be defined on any set, while complex fuzzy ideals are specifically defined on algebraic structures such as BCK/BCI-algebras.
3. Closure Properties:
   - Both complex fuzzy sets and complex fuzzy ideals exhibit closure properties, but in different contexts. Complex fuzzy sets may not necessarily maintain closure properties under operations defined on the underlying algebraic structure, whereas complex fuzzy ideals typically maintain closure properties under the algebraic operations of the BCK/BCI-algebras.
4. Applications:
   - Complex fuzzy sets have been applied in various fields, including decision making, pattern recognition, and control systems.
   - Complex fuzzy ideals are particularly useful in algebraic structures for analyzing the behavior of operations in the presence of uncertainty.
5. Complexity and Analysis:
   - Complex fuzzy sets, while still complex, are easier to analyze in some cases because they are not constrained by algebraic structure operations.

- Complex fuzzy ideals may involve more intricate mathematical analysis due to their algebraic nature and the interplay of complex numbers with algebraic operations.

6. Advantages

1. Complex fuzzy ideals generalize classical fuzzy ideals in algebraic structures such as BCK/BCI-algebras.
2. BCK/BCI-algebras with complex fuzzy ideals can adapt to uncertain environments by dynamically adjusting the degree of membership of the elements to the ideals.
3. Complex fuzzy ideals are naturally integrated into fuzzy logic systems, allowing seamless reasoning and inference within the context of BCK/BCI-algebras.
4. Complex fuzzy ideals preserve the closure of algebraic operations and ensure consistency with the underlying algebraic structure.
5. Complex fuzzy ideals facilitate algebraic analysis tasks such as congruence relations and lattice structures and enable deeper investigation of algebraic properties.

7. Conclusions

The expansion of crisp sets to fuzzy sets in terms of membership functions is analogous to the expansion of integers to real numbers. Just as integers $(\mathbb{Z})$ were extended to the real number line $(\mathbb{R})$, the range of membership functions expands from $\{0, 1\}$ to the unit interval $[0, 1]$. According to this historical progression, the development of sets of numbers did not end with real numbers. This process continued with the introduction of complex numbers $(\mathbb{C})$. Likewise, this extension could lead to further advances in fuzzy set theory.

This article contributes to the concept of complex fuzzy sets characterized by membership functions extending beyond $[0, 1]$ to the unit circle in the complex plane. Ramot et al. [16,17] presented this in their paper, which features a membership function with values in the complex domain.

In this work, we used complex fuzzy sets to obtain the generalization of fuzzy set theory in BCK/BCI-algebras. We introduced the notion of a complex fuzzy subalgebra in a BCK/BCI-algebra and examined related properties. We have defined and studied the modal and level operators of complex fuzzy subalgebras in BCK/BCI-algebras. We have studied various operations and the laws of a complex fuzzy system, including union, intersection, complement, and simple and bounded differences of complex fuzzy ideals in BCK/BCI-algebras. We have provided our proposed approach in the form of an algorithm.

Future work could explore further real-world applications of complex fuzzy sets, and linear complex Diophantine fuzzy sets could also be investigated. Furthermore, we will introduce complex intuitionistic fuzzy ideals in BCK-algebras with applications in the TOPSIS, Electure-I, and Electure-II methods using multi-criteria decision problems.

Author Contributions: Conceptualization, M.B., T.R., A.A.-M. and K.A.; methodology, M.B.; validation, K.A. and A.A.-M.; formal analysis, K.A. and T.R.; investigation, A.A.-M.; writing—original draft preparation, M.B.; writing—review and editing, K.A. and A.A.-M. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Data are contained within the article.

Acknowledgments: The Researchers would like to thank the Deanship of Graduate Studies and Scientific Research at Qassim University for financial support (QU-APC-2024-9/1).

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A

The following, we show some algorithms for studying the structure of complex fuzzy subalgebras and ideals in BCK/BCI-algebras using Definition 4 and definition 9.

Appendix A.1. Algorithm for Complex Fuzzy Subalgebras in BCK/BCI-Algebras

def 4 is_complex_fuzzy_subalgebra($U, \diamond, \tilde{Ą}$):
stop = False
i = 1
while i $\leq |U|$ and not stop:
if $\bar{\upsilon}_{\tilde{Ą}}(0) < \bar{\upsilon}_{\tilde{Ą}}(\check{\varsigma}_i)$:
stop = True
i += 1
j = 1
while j $\leq |U|$ and not stop:
if $\bar{\upsilon}_{\tilde{Ą}}(\check{\varsigma}_i \diamond \check{\varrho}_j) < \bar{\upsilon}_{\tilde{Ą}}(\check{\varsigma}_i) \wedge \bar{\upsilon}_{\tilde{Ą}}(\check{\varrho}_j)$:
stop = True
j += 1
if stop:
print("$\tilde{Ą}$ is not a complex fuzzy subalgebra of U")
else:
print("$\tilde{Ą}$ is a complex fuzzy subalgebra of U")

Appendix A.2. Algorithm for Complex Fuzzy Ideals in BCK/BCI-Algebras

def 9 is_complex_fuzzy_ideal($U, \diamond, \tilde{Ą}$):
stop = False
i = 1
while i $\leq |U|$ and not stop:
if $\bar{\upsilon}_{\tilde{Ą}}(0) < \bar{\upsilon}_{\tilde{Ą}}(\check{\varsigma}_i)$:
stop = True
i += 1
j = 1
while j $\leq |U|$ and not stop:
if $\bar{\upsilon}_{\tilde{Ą}}(\check{\varsigma}_i) < \bar{\upsilon}_{\tilde{Ą}}(\check{\varsigma}_i \diamond \check{\varrho}_j) \wedge \bar{\upsilon}_{\tilde{Ą}}(\check{\varrho}_j)$:
stop = True
j += 1
if stop:
print("$\tilde{Ą}$ is not a complex fuzzy ideal of U")
else:
print("$\tilde{Ą}$ is a complex fuzzy ideal of U")

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