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Tractability of Multivariate Approximation Problem on Euler and Wiener Integrated Processes

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Abstract: This paper examines the tractability of multivariate approximation problems under the normalized error criterion for a zero-mean Gaussian measure in an average-case setting. The Gaussian measure is associated with a covariance kernel, which is represented by the tensor product of one-dimensional kernels corresponding to Euler and Wiener integrated processes with non-negative and nondecreasing smoothness parameters $\{r_d\}_{d\in\mathbb{N}}$. We give matching sufficient and necessary conditions for various concepts of tractability in terms of the asymptotic properties of the regularity parameters, except for $(s,0)$-WT.

Keywords: Euler and Wiener integrated processes; tractability; normalized error criterion; average-case setting

MSC: 41A63; 65Y20; 68Q25

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1. Introduction

The tractability of multivariate problems $S = \{S_d\}_{d\in\mathbb{N}}$ has become a very active research field (see [1–3]), with many scholars devoted to studying the behavior of the information complexity $n(\varepsilon, S_d)$, which changes as the variable ε tends to zero and d goes to infinity. As before, we define information complexity as the minimal number of continuous linear functionals needed to seek an ε-approximation of the operator $S_d : F_d \to G_d$, and the considered problems are related to a zero-mean Gaussian measure under the normalized error criterion and in an average-case setting. Note that G_d is a Hilbert space and that F_d is a Banach space equipped with Gaussian measure μ_d with a zero mean value. The algorithm $A : F_d \to G_d$ is considered an $\varepsilon-$approximation of S_d if

$$\left(\int_{F_d} \|S_d(f) - A(f)\|^2_{G_d}\mu_d(df)\right)^{1/2} \le \varepsilon\left(\int_{F_d} \|S_d(f)\|^2_{G_d}\mu_d(df)\right)^{1/2}.$$

The tractability concepts on multivariate problems were first proposed in 1994 by professor H.Woźni-kowski (see [4]). In general, a problem is considered intractable if $n(\varepsilon, S_d)$ is an exponential function of the variable ε^{-1} or d. Otherwise, it is considered tractable. Until now, various tractability concepts have been studied for many multidimensional approximation problems in different error settings. Among these numerous studies, multivariate approximation and integration are the most extensively studied and important issues. In brief, we now recall some of the basic tractability concepts (see [1,5–7]).

For multivariate problems $S = \{S_d\}_{d\in\mathbb{N}}$, we state the following:

- Strong polynomial tractability (SPT) holds if there are non-negative numbers C and p such that
$$n(\varepsilon, S_d) \le C(\varepsilon^{-1})^p \qquad \text{for all } d \in \mathbb{N},\ \varepsilon \in (0,1).$$
The infimum of all p, for which the above inequality holds, is defined as the exponent p^* of SPT.

- Polynomial tractability (PT) holds if there are non-negative numbers C, p, and q such that
$$n(\varepsilon, S_d) \le C d^q (\varepsilon^{-1})^p \qquad \text{for all } d \in \mathbb{N},\ \varepsilon \in (0,1).$$
- Quasi-polynomial tractability (QPT) holds if there are positive numbers C and t such that
$$n(\varepsilon, S_d) \le C \exp(t(1+\ln d)(1+\ln \varepsilon^{-1})) \qquad \text{for all } d \in \mathbb{N},\ \varepsilon \in (0,1).$$
- Uniform weak tractability (UWT) holds if, for all $\alpha, \beta > 0$,
$$\lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln n(\varepsilon, S_d)}{\varepsilon^{-\alpha} + d^{\beta}} = 0,$$
which is equivalent to
$$\lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln n(\varepsilon, S_d)}{\varepsilon^{-\alpha} + d^{\alpha}} = 0, \quad \text{for any } \alpha > 0. \tag{1}$$
- Weak tractability (WT) holds if
$$\lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln n(\varepsilon, S_d)}{\varepsilon^{-1} + d} = 0.$$
- (s,t)-weak tractability ((s,t)-WT) holds for some non-negative s and t if
$$\lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln n(\varepsilon, S_d)}{\varepsilon^{-s} + d^{t}} = 0. \tag{2}$$

Based on the above tractability definition, we can easily obtain the following logical relation:
$$\text{SPT} \Rightarrow \text{PT} \Rightarrow \text{QPT} \Rightarrow \text{UWT} \Rightarrow \text{WT}.$$

We note that many papers have studied the various concepts of tractability on the approximation of integrated Euler and Wiener processes (see [8–13]). Usually, the covariance kernel of the Gaussian measure used in these papers is the tensor product of the one-dimensional kernels corresponding to these random processes with nondecreasing and non-negative smoothness parameters $\{r_d\}_{d\in\mathbb{N}}$. In this regard, we investigate a more special case of the covariance kernel (for more details, see Section 2), which is essentially different to that in previous papers. We finally obtain matching necessary and sufficient conditions such that the above concepts of tractability hold, and the proofs employ several techniques and methods for the general covariance kernel. It should be noted that $(0,t)$-WT and $(s,0)$-WT have not been discussed before. In this regard, we prove that the considered multivariate problem is not $(0,t)$-WT. For $(s,0)$-WT, we provide a sufficient condition, and whether this condition is a necessary condition for matching remains an open question.

Following is an outline of this paper. We first provide some basic concepts and background information about the multivariate problem of the integrated Euler and Wiener processes in Section 2. The proofs of our main results are given and proven in Sections 3 and 4.

2. Euler and Wiener Integrated Processes

In the following, we use $\mathbb{N}$, $\mathbb{N}_0$, and $\mathbb{R}$ to represent the sets of positive integers, non-negative integers, and real numbers, respectively. Furthermore, we denote their $d-$ary Cartesian powers as $\mathbb{N}^d$, $\mathbb{N}_0^d$, and $\mathbb{R}^d$ for each $d \in \mathbb{N}$. Additionally, we define $\ln^+ x = \max\{1, \ln x\}$ for $x > 0$.

A linear multivariate problem is defined as a sequence $S = \{S_d\}_{d\in\mathbb{N}}$ of continuous linear operators $S_d : F_d \to G_d$, where G_d is a Hilbert space, and F_d is a separable Banach space equipped with Gaussian measure μ_d with a zero mean value.

Now, let $\nu_d = \mu_d S_d^{-1}$ be a Gaussian measure with a zero mean value induced on space G_d by operator S_d and measure μ_d on F_d. On the other side, let $C_{\nu_d} : G_d \to G_d$ be the covariance operator of measure ν_d. Then, C_{ν_d} becomes a self-adjoint, non-negative definite operator and has finite trace (see [1]). The eigenpairs of C_{ν_d} are defined as $\{(\lambda_{d,j}, \eta_{d,j})\}_{j\in\mathbb{N}}$, which satisfy

$$\lambda_{d,1} \geq \lambda_{d,2} \geq \cdots \geq \lambda_{d,j} \cdots .$$

For each $d \in \mathbb{N}$, we use information-based algorithms

$$A_{n,d}(f) = \phi_{n,d}(L_1(f), L_2(f), \ldots, L_n(f)) \tag{3}$$

to approximate $S_d(f)$, where $f \in F_d$, $\phi_{n,d} : \mathbb{R}^n \to G_d$ is an arbitrary measurable mapping, and $L_1, L_2, \ldots, L_n$ are continuous linear functionals on F_d. As a special case, we define $A_{0,d} = 0$.

The average-case approximation error for the algorithm $A_{n,d}$ of the form (3) is defined by

$$e(A_{n,d}) := \left(\int_{F_d} \| S_d(f) - A_{n,d}(f) \|_{G_d}^2 \, \mu_d(df) \right)^{1/2}.$$

Then, the n-th minimal average-case error for $n \geq 1$ is given by

$$e(n,d) = \inf_{A_{n,d}} e(A_{n,d}) = \left(\sum_{j=n+1}^{\infty} \lambda_{d,j} \right)^{1/2},$$

where the infimum is taken over by all algorithms of the form (3). In fact, this can be achieved by the n-th optimal algorithm

$$A_{n,d}^*(f) = \sum_{j=1}^{n} \langle S_d f, \eta_{d,j} \rangle_{G_d} \eta_{d,j}.$$

For $n = 0$, it is easy to see that $A_{0,d}^* = 0$ and the *initial error* $e(0,d)$ is given by (see [1])

$$e(0,d) = \left(\int_{F_d} \| S_d(f) \|_{G_d}^2 \, \mu_d(df) \right)^{1/2} = \left(\sum_{j=1}^{\infty} \lambda_{d,j} \right)^{1/2}.$$

Using the above preparation knowledge, we give the definition of information complexity. For $\varepsilon \in (0,1)$ and the absolute or normalized error criterion, the information complexity of S_d is defined as

$$n^{\mathrm{X}}(\varepsilon, S_d) = \min\{n \in \mathbb{N} : e(n,d) \leq \varepsilon \mathrm{CRI}_d\},$$

where

$$\mathrm{CRI}_d = 1 \text{ and X = abs for the absolute error criterion,}$$

$$\mathrm{CRI}_d = e(0,d) \text{ and X = nor for the normalized error criterion.}$$

Hence, the information complexity with the normalized error criterion can be expressed by

$$n^{\mathrm{nor}}(\varepsilon, S_d) = \min\left\{ n \in \mathbb{N} : \sum_{j=n+1}^{\infty} \lambda_{d,j} \leq \varepsilon^2 \sum_{j=1}^{\infty} \lambda_{d,j} \right\}, \tag{4}$$

while, for the absolute error criterion,

$$n^{\mathrm{abs}}(\varepsilon, S_d) = \min\left\{ n \in \mathbb{N} : \sum_{j=n+1}^{\infty} \lambda_{d,j} \leq \varepsilon^2 \right\}. \tag{5}$$

Now, we introduce the approximation problems $\mathrm{APP} = \{\mathrm{APP}_d\}_{d\in\mathbb{N}}$,

$$\mathrm{APP}_d : C([0,1]^d) \to L_2([0,1]^d) : f \mapsto f.$$

It should be noted that the continuous real function space $C([0,1]^d)$ is equipped with Gaussian measure μ_d with a zero mean value, and its covariance kernel corresponds to two random processes, i.e., Euler and Wiener integrated process. In the following, we provide basic knowledge on and detail the important properties of Euler and Wiener integrated processes. For $t \in [0,1]$, let $W(t)$ be a standard Wiener process, i.e., a Gaussian measure with a zero mean value and its covariance kernel

$$K^E_{1,0}(s,t) = K^W_{1,0}(s,t) := \min(s,t).$$

For $r = 0,1,2,\ldots$, the two sequences of random processes X^E_r and X^W_r in the interval $[0,1]$ are recursively defined on parameter r by $X^E_0 = X^W_0 = W$ and

$$X^E_{r+1}(t) = \int_{1-t}^{1} X^E_r(s)ds,$$

$$X^W_{r+1}(t) = \int_0^t X^W_r(s)ds.$$

Usually, we refer to $\{X^E_r\}_{r\in\mathbb{N}_0}$ as the univariate Euler integrated process and $\{X^W_r\}_{r\in\mathbb{N}_0}$ as the univariate integrated Wiener process.

The Gaussian measure corresponding to random processes X^E_r and X^W_r is focused on a series of functions that are r times continuously differentiable but have different boundary conditions. For the covariance kernel of X^E_r, it is represented by

$$K^E_{1,r}(x,y) = \int_{[0,1]^r} \min(x,s_1)\min(s_1,s_2)\cdots\min(s_r,y)ds_1ds_2\cdots ds_r,$$

usually referred to as the Euler kernel. Furthermore, this kernel can be expressed by Euler polynomials (see [8]). For the covariance kernel of X^W_r, it is denoted as

$$K^W_{1,r}(x,y) = \int_0^{\min(x,y)} \frac{(x-u)^r}{r!}\frac{(y-u)^r}{r!}du,$$

called the Wiener kernel.

For $\mathbf{x},\mathbf{y} \in [0,1]^d$, the corresponding tensor product kernels are represented by

$$K^E_d(\mathbf{x},\mathbf{y}) = \prod_{k=1}^d K^E_{1,r_d}(x_k,y_k) \text{ and } K^W_d(\mathbf{x},\mathbf{y}) = \prod_{k=1}^d K^W_{1,r_d}(x_k,y_k),$$

with a sequence of nondecreasing, non-negative integers $\{r_k\}_{k\in\mathbb{N}}$, and

$$0 \le r_1 \le r_2 \le r_3 \cdots. \tag{6}$$

We note that the tensor product kernels K^E_d and K^W_d are essentially different to those in [8,9,11,12]. For the multivariate problem APP, paper [14] obtained the eigenvalues of the covariance operators of the induced measure corresponding to the above two random processes, i.e.,

$$\{\lambda^X_{d,j}\}_{j\in\mathbb{N}} = \{\lambda^X_{r_d}(j_1)\lambda^X_{r_d}(j_2)\cdots\lambda^X_{r_d}(j_d)\}_{(j_1,\ldots,j_d)\in\mathbb{N}^d}, \quad X \in \{E,W\},$$

where

$$\lambda^E_{r_d}(j) = \left(\frac{1}{\pi(j-\frac{1}{2})}\right)^{2r_d+2}, \tag{7}$$

for all $j \in \mathbb{N}$, and

$$\lambda_{r_d}^W(j) = \left(\frac{1}{\pi(j-\frac{1}{2})}\right)^{2r_d+2} + \mathcal{O}(j^{-(2r_d+3)}), \ \ j \to \infty, \tag{8}$$

where, for $f, g : \mathbb{N} \to [0, \infty)$, $f(k) = \mathcal{O}(g(k))$ as $k \to \infty$ implies that there exists $C > 0$ and $k_0 \in \mathbb{N}$ such that $f(k) \le Cg(k)$ for any $k \ge k_0$.

Note that, for all $d \in \mathbb{N}$, $X \in \{E, W\}$,

$$\sum_{j=1}^{\infty} (\lambda_{d,j}^X)^\tau = \left(\sum_{j=1}^{\infty} (\lambda_{r_d}^X(j))^\tau\right)^d, \ \ \forall\, \tau > 0, \tag{9}$$

and

$$h_d := \frac{\lambda_{r_d}^E(2)}{\lambda_{r_d}^E(1)} = \frac{1}{3^{2r_d+2}}.$$

It is proven in [8] that

$$\lambda_r^W(1) = \frac{1}{(r!)^2}\left(\frac{1}{(2r+2)(2r+1)} + \mathcal{O}(r^{-4})\right), \ \ r \to \infty, \tag{10}$$

$$\lambda_r^W(2) = \Theta\left(\frac{1}{(r!)^2 r^4}\right), \ \ r \to \infty, \tag{11}$$

$$\sup_{\tau \in [\tau_0, 1]} \frac{\sum_{j=3}^{\infty} (\lambda_r^W(j))^\tau}{(\lambda_r^W(2))^\tau} = \mathcal{O}(r^{-h}) \ \ \text{for some } h > 0 \text{ and for all } \tau_0 \in \left(\frac{3}{5}, 1\right], \tag{12}$$

where, for $f, g : \mathbb{N} \to [0, \infty)$, $g(k) = \Theta(f(k))$ as $k \to \infty$ implies that $f(k) = \mathcal{O}(g(k))$ and $g(k) = \mathcal{O}(f(k))$ as $k \to \infty$. Furthermore, from (10) and (11), one has

$$Q_d := \frac{\lambda_r^W(2)}{\lambda_r^W(1)} = \Theta(r^{-2}) = \Theta((1+r)^{-2}), \ \ r \to \infty. \tag{13}$$

Detailed information about the multivariate approximation of integrated Euler and Wiener processes is provided in [8].

Remark 1. *Let $\overline{APP} = \{\overline{APP}_d\}_{d \in \mathbb{N}}$ be an approximation problem, and the eigenvalues of the covariance operators are*

$$\left\{\overline{\lambda}_{d,j}^X\right\}_{j \in \mathbb{N}} = \left\{\lambda^X(j_1, d)\lambda^X(j_2, d)\cdots\lambda^X(j_d, d)\right\}_{(j_1, j_2, \dots, j_d) \in \mathbb{N}^d}, \ \ X \in \{E, W\},$$

where and in the following

$$\lambda^X(j, d) := \frac{\lambda_{r_d}^X(j)}{\lambda_{r_d}^X(1)}, \ \ j \in \mathbb{N}.$$

Hence, from (9) and (4), we know that $n^{nor}(\varepsilon, \overline{APP}_d) = n^{nor}(\varepsilon, APP_d)$, and then $\overline{APP}$ and APP have the same tractability properties for the normalized error criterion. Specially, for $X = E$,

$$\lambda^E(j, d) = (2j-1)^{-2(r_d+1)}, \ \ j \in \mathbb{N}. \tag{14}$$

3. Tractability of Euler Integrated Process

In this section, we study the various concepts of tractability on the Euler integrated process and give matching necessary and sufficient conditions, except for $(s, 0)$-WT.

Theorem 1. *Consider the multivariate approximation problem APP for the Euler integrated process. Then, for the normalized error criterion,*

(i) *SPT holds if PT holds if*

$$S_\tau := \sup_{k\in\mathbb{N}} 3^{-2\tau r_k} k < \infty \text{ for some } \tau \in (0,1),$$

or equivalently if

$$A := \liminf_{k\to\infty} \frac{r_k}{\ln k} > \frac{1}{2\ln 3}.$$

If so, then the exponent of SPT is

$$p^* = \max\left(\frac{2}{2r_1+1}, \frac{2}{2A\ln 3 - 1}\right). \tag{15}$$

(ii) *QPT holds if*

$$\sup_{d\in\mathbb{N}} \frac{d(1+r_d)3^{-2r_d}}{\ln^+ d} < \infty. \tag{16}$$

(iii) *UWT holds if*

$$A := \liminf_{k\to\infty} \frac{r_k}{\ln k} \geq \frac{1}{2\ln 3}.$$

(iv) *(s,t)-WT with $s > 0$ and $t > 1$ always holds.*

(v) *(s,1)-WT with $s > 0$ holds if WT holds if*

$$\lim_{k\to\infty} r_k = \infty. \tag{17}$$

(vi) *(s,t)-WT holds with $s > 0$ and $t \in (0,1)$ if*

$$\lim_{k\to\infty} k^{(1-t)} 3^{-2r_k}(r_k+1) = 0. \tag{18}$$

(vii) *APP is not $(0,t)$-WT with $t > 0$.*

(viii) *If APP is $(s,0)$-WT with $s > 0$, then (18) holds with $t = 0$. However, if*

$$\lim_{k\to\infty} \frac{r_k}{\ln k} = \infty, \tag{19}$$

then APP is $(s,0)$-WT with $s > 0$.

Proof of part (i). Based on the logical relationship between SPT and PT, we can easily observe that, to prove (i), it is enough to show

$$A > \frac{1}{2\ln 3} \Rightarrow S_\tau < \infty \Rightarrow \text{SPT} \Rightarrow \text{ PT} \Rightarrow A > \frac{1}{2\ln 3}. \tag{20}$$

We first prove $A > 1/(2\ln 3) \Rightarrow S_\tau < \infty$ for some $\tau \in (0,1)$. Indeed, let $A > 1/(2\ln 3)$, and then, for some $\delta > 0$, there exists a positive integer k_δ such that $r_k/\ln k > (1+\delta)/(2\ln 3)$ when $k \geq k_\delta$. Therefore, $3^{-2\tau r_k} < k^{-(1+\delta)\tau}$ and, thus, $S_\tau < \infty$ whenever $1/(1+\delta) < \tau < 1$. Note that, in this case, $1+\delta \leq 2A\ln 3$, which yields $\tau > 1/(1+\delta) \geq 1/(2A\ln 3)$.

Now, we turn to prove that $S_\tau < \infty \Rightarrow$ SPT. By Chapter 6 in [1], we know that PT holds if there exist $q \geq 0$ and $\tau \in (0,1)$ such that

$$C := \sup_{d\in\mathbb{N}} \frac{\left(\sum_{j=1}^\infty \lambda_{d,j}^\tau\right)^{1/\tau}}{\sum_{j=1}^\infty \lambda_{d,j}} d^{-q} < \infty. \tag{21}$$

Furthermore, APP is SPT if (21) holds with $q = 0$, and the exponent of SPT is given by

$$p^* = \inf\left\{ \frac{2\tau}{1-\tau} \,\middle|\, \tau \text{ satisfies (21) with } q = 0 \right\}. \tag{22}$$

Now and in the remainder of this section, we let $\lambda_{d,j} = \lambda^E_{d,j}$. Take $\tau \in (0,1)$; then, by (7) and (9) , we have

$$\frac{\left(\sum_{j=1}^{\infty} \lambda_{d,j}^{\tau}\right)^{1/\tau}}{\sum_{j=1}^{\infty} \lambda_{d,j}} = \frac{\left(\sum_{j=1}^{\infty} \lambda_{r_d}^{\tau}(j)\right)^{d/\tau}}{\left(\sum_{j=1}^{\infty} \lambda_{r_d}(j)\right)^{d}} = \frac{\left(1 + \sum_{j=2}^{\infty} (2j-1)^{-2\tau(r_d+1)}\right)^{d/\tau}}{\left(1 + \sum_{j=2}^{\infty} (2j-1)^{-2(r_d+1)}\right)^{d}}.$$

It follows from the proof of Theorem 1 in [8] that, for all $x \in (1/(2r_1+2), 1]$,

$$3^{-2x(r_d+1)} \le \sum_{j=2}^{\infty} (2j-1)^{-2x(r_d+1)} \le \frac{2x(r_1+1)+2}{2x(r_1+1)-1} 3^{-2x(r_d+1)}. \tag{23}$$

Therefore, for $\tau \in (1/(2r_1+2), 1)$,

$$\frac{\left(\sum_{j=1}^{\infty} \lambda_{d,j}^{\tau}\right)^{1/\tau}}{\sum_{j=1}^{\infty} \lambda_{d,j}} = \frac{\left(1 + a_d 3^{-2\tau(r_d+1)}\right)^{d/\tau}}{(1 + b_d 3^{-2(r_d+1)})^d}, \tag{24}$$

where $a_d \ge b_d$, and both are uniformly bounded,

$$1 \le a_d \le \frac{2\tau(r_1+1)+2}{2\tau(r_1+1)-1} \text{ and } 1 \le b_d \le \frac{2r_1+4}{2r_1+1}. \tag{25}$$

Now, we assume that $S_\tau < \infty$ for some $\tau \in (1/(2r_1+2), 1)$. By combining (24) and (25), we have

$$\begin{aligned}\sup_{d\in\mathbb{N}} \frac{\left(\sum_{j=1}^{\infty} \lambda_{d,j}^{\tau}\right)^{1/\tau}}{\sum_{j=1}^{\infty} \lambda_{d,j}} &\le \sup_{d\in\mathbb{N}} \left(1 + a_d 3^{-2\tau(r_d+1)}\right)^{d/\tau} \\ &\le \exp\left(\tau^{-1} \sup_{d\in\mathbb{N}} a_d d 3^{-2\tau(r_d+1)}\right) \le \exp\left(\tau^{-1} \frac{2\tau(r_1+1)+2}{2\tau(r_1+1)-1} \cdot S_\tau\right) < \infty,\end{aligned}$$

where we use the fact that $1 + x \le e^x$ for $x \ge 0$. Therefore, (21) holds with $q = 0$, and we conclude that $S_\tau < \infty$ implies that SPT holds. Furthermore, from the above proof, we know that, if $\tau > \max\{\frac{1}{2r_1+2}, \frac{1}{2A\ln 3}\}$, then (21) holds with $q = 0$. Thus, (22) implies that

$$p^* \le \max\left(\frac{2}{2r_1+1}, \frac{2}{2A\ln 3 - 1}\right). \tag{26}$$

The relation on SPT$\Rightarrow$PT is trivial.

We now suppose that PT holds. From (21) and (24), we have

$$\frac{\left(1 + a_d 3^{-2\tau(r_d+1)}\right)^{d/\tau}}{(1 + b_d 3^{-2(r_d+1)})^d} \le C d^q$$

for some $C, q \geq 0$ and $\tau \in (1/(2r_1+2), 1)$. By $a_d \geq b_d$, one has

$$\frac{\left(1 + a_d 3^{-2\tau(r_d+1)}\right)^{1/\tau}}{1 + b_d 3^{-2(r_d+1)}} - 1 \geq \frac{\left(1 + a_d 3^{-2\tau(r_d+1)}\right) - (1 + b_d 3^{-2(r_d+1)})}{1 + b_d 3^{-2(r_d+1)}}$$
$$\geq \frac{a_d (1 - 3^{-2(r_d+1)(1-\tau)}) 3^{-2\tau(r_d+1)}}{1 + b_d 3^{-2(r_d+1)}} = c_d 3^{-2\tau(r_d+1)},$$

where, based on (25) and (6), we know that

$$c_d = \frac{a_d (1 - 3^{-2(r_d+1)(1-\tau)})}{1 + b_d 3^{-2(r_d+1)}} \geq \frac{1 - 3^{-2(r_1+1)(1-\tau)}}{1 + \frac{2r_1+4}{2r_1+1} \cdot \frac{1}{3^2}} =: B \in (0,1).$$

Hence, we obtain

$$(1 + B 3^{-2\tau(r_d+1)})^d \leq C d^q.$$

By taking logarithms on the above relation, one has

$$(\ln 2) B d 3^{-2\tau(r_d+1)} \leq d \ln(1 + B 3^{-2\tau(r_d+1)}) \leq \ln C + q \ln^+ d,$$

where, in the first inequality above, we adopt the fact that $\ln(1+x) \geq x \ln 2$ for $0 \leq x \leq 1$. Therefore, this means that

$$M := \sup_{d \in \mathbb{N}} \frac{d}{\ln^+ d} 3^{-2\tau(r_d+1)} < \infty.$$

Then, from the above inequality, we can easily obtain $d \cdot 3^{-2\tau(r_d+1)} \leq M \ln^+ d$. By taking the logarithms of this inequality, we obtain

$$\frac{r_d + 1}{\ln d} \geq \frac{1 - \frac{\ln \ln^+ d + \ln M}{\ln d}}{2\tau \ln 3} \quad \text{for all } d \geq 2,$$

and this clearly implies that

$$A \geq 1/(2\tau \ln 3) > 1/(2 \ln 3), \tag{27}$$

as claimed. Therefore, all statements in (20) are equivalent. We also notice that the first inequality in (27) implies that $\tau \geq 1/(2A \ln 3)$. Moreover, it is obvious to see that (21) holds only if $\tau > 1/(2r_1+2)$. It follows from (22) that

$$p^* \geq \max\left(\frac{2}{2r_1+1}, \frac{2}{2A \ln 3 - 1}\right). \tag{28}$$

Thus, by (26) and (28), we obtain (15). The proof of (i) is complete. □

Proof of part (ii). From Theorem 2 in [10], we know that APP is QPT if there exists $\delta \in (0,1)$ such that

$$\sup_{d \in \mathbb{N}} \frac{\sum_{j=1}^{\infty} \lambda_{d,j}^{1-\delta/\ln^+ d}}{\left(\sum_{j=1}^{\infty} \lambda_{d,j}\right)^{1-\delta/\ln^+ d}} < \infty. \tag{29}$$

Regarding sufficiency, we first prove that (16) implies (29) with $\delta = 1/2$. From (9), we have

$$\sup_{d \in \mathbb{N}} \frac{\sum_{j=1}^{\infty} \lambda_{d,j}^{1-\frac{1}{2\ln^+ d}}}{\left(\sum_{j=1}^{\infty} \lambda_{d,j}\right)^{1-\frac{1}{2\ln^+ d}}} = \sup_{d \in \mathbb{N}} \frac{\left(\sum_{j=1}^{\infty} \lambda(j,d)^{1-\frac{1}{2\ln^+ d}}\right)^d}{\left(\sum_{j=1}^{\infty} \lambda(j,d)\right)^{d-\frac{d}{2\ln^+ d}}},$$

where $\lambda(j,d) = \lambda^E(j,d)$. We now divide the last product into two products

$$\Pi_1(d) := \left(\sum_{j=1}^{\infty} \lambda(j,d)\right)^{\frac{d}{2\ln^+ d}}, \ \ \Pi_2(d) := \left(\frac{\sum_{j=1}^{\infty} \lambda(j,d)^{1-\frac{1}{2\ln^+ d}}}{\sum_{j=1}^{\infty} \lambda(j,d)}\right)^{d}.$$

For $\Pi_1(d)$, by (14) and (23) , we see that, for $x = 1$, $\sum_{j=2}^{\infty} \lambda(j,d) \le (2r_1+4)/(2r_1+1)\lambda(2,d) < 4\lambda(2,d)$; then, we have

$$\begin{aligned}\Pi_1(d) &= \left(1 + \sum_{j=2}^{\infty} \lambda(j,d)\right)^{\frac{d}{2\ln^+ d}} \le \exp\left(\frac{d}{2\ln^+ d} \sum_{j=2}^{\infty} \lambda(j,d)\right) \\ &\le \exp\left(\frac{2d}{\ln^+ d}\lambda(2,d)\right) = \exp\left(\frac{2d}{\ln^+ d} 3^{-2(r_d+1)}\right).\end{aligned}$$

Clearly, (16) implies that $\sup_{d\in\mathbb{N}} \Pi_1(d) < \infty$.

We now consider the product $\Pi_2(d)$. From the proof of Theorem 1 in [8], we know that there exists $C > 0$ such that

$$\frac{\sum_{j=1}^{\infty} \lambda(j,d)^{1-\frac{1}{2\ln^+ d}}}{\sum_{j=1}^{\infty} \lambda(j,d)} \le \exp\left(d^{-\frac{3}{2}} + \frac{C\lambda(2,d)|\ln \lambda(2,d)|}{\ln^+ d} + Cd^{-\frac{\ln 5}{\ln 3}}\right).$$

Therefore, we easily obtain

$$\begin{aligned}\Pi_2(d) &\le \exp\left(d\left(d^{-\frac{3}{2}} + \frac{C\lambda(2,d)|\ln \lambda(2,d)|}{\ln^+ d} + Cd^{-\frac{\ln 5}{\ln 3}}\right)\right) \\ &\le \exp\left(d^{-\frac{1}{2}} + \frac{Cd \cdot 3^{-2r_d}(r_d+1)}{\ln^+ d} + Cd^{1-\frac{\ln 5}{\ln 3}}\right),\end{aligned}$$

and, by (16), we find that $\sup_{d\in\mathbb{N}} \Pi_2(d) < \infty$. Therefore,

$$\sup_{d\in\mathbb{N}} \Pi_1(d)\Pi_2(d) \le \sup_{d\in\mathbb{N}} \Pi_1(d) \sup_{d\in\mathbb{N}} \Pi_2(d) < \infty,$$

which implies that (29) holds. Hence, APP is QPT.

Regarding necessity, assume that QPT holds. Now, we let

$$\Lambda(r_d) := \sum_{j=1}^{\infty} \lambda_{r_d}(j),$$

and then, by (9), one has

$$\Lambda_d := \sum_{j=1}^{\infty} \lambda_{d,j} = (\Lambda(r_d))^d.$$

Using the proof of (20) in [10], we obtain

$$\sum_{j=1}^{\infty} \lambda_{d,j} \ln \lambda_{d,j} = d\left(\sum_{j=1}^{\infty} \lambda_{r_d}(j) \ln \lambda_{r_d}(j)\right) \frac{\Lambda_d}{\Lambda(r_d)}.$$

Then, by the above equality, one has

$$\begin{aligned}\sum_{j=1}^{\infty}\frac{\lambda_{d,j}}{\Lambda_d}\ln\left(\frac{\Lambda_d}{\lambda_{d,j}}\right) &= \ln\Lambda_d-\Lambda_d^{-1}\sum_{j=1}^{\infty}\lambda_{d,j}\ln\lambda_{d,j} \\ &= d\ln\Lambda(r_d)-d\sum_{j=1}^{\infty}\frac{\lambda_{r_d}(j)}{\Lambda(r_d)}\ln\lambda_{r_d}(j) \\ &= d\sum_{j=1}^{\infty}\frac{\lambda_{r_d}(j)}{\Lambda(r_d)}\ln\left(\frac{\Lambda(r_d)}{\lambda_{r_d}(j)}\right) \\ &= d\sum_{j=1}^{\infty}\frac{\lambda(j,d)}{\Lambda(d)}\ln\left(\frac{\Lambda(d)}{\lambda(j,d)}\right), \end{aligned} \tag{30}$$

where $\Lambda(d) := \sum_{j=1}^{\infty}\lambda(j,d)$. From Corollary 4 in [10], we know that, if quasi-polynomial tractability holds, then

$$\sup_{d\in\mathbb{N}}\frac{1}{\ln^+ d}\sum_{j=1}^{\infty}\frac{\lambda_{d,j}}{\Lambda_d}\ln\left(\frac{\Lambda_d}{\lambda_{d,j}}\right)<\infty. \tag{31}$$

Since APP is QPT, by combining (30) and (31) , we obtain

$$\sup_{d\in\mathbb{N}}\frac{d}{\ln^+ d}\sum_{j=1}^{\infty}\frac{\lambda(j,d)}{\Lambda(d)}\ln\left(\frac{\Lambda(d)}{\lambda(j,d)}\right)<\infty. \tag{32}$$

Since $\Lambda(d)/\lambda(j,d)>1$, it is easy to see that all terms in the sums over j are positive. By omitting all terms for $j\neq 2$ in the last condition, we have

$$\sup_{d\in\mathbb{N}}\frac{d}{\ln^+ d}\frac{\lambda(2,d)}{\Lambda(d)}\ln\left(\frac{\Lambda(d)}{\lambda(2,d)}\right)<\infty. \tag{33}$$

Furthermore, due to $1<\Lambda(d)\le\Lambda(1)$, we obtain

$$\sup_{d\in\mathbb{N}}\frac{d}{\ln^+ d}\lambda(2,d)\ln\left(\frac{1}{\lambda(2,d)}\right)<\infty.$$

Obviously, the above condition is equivalent to (16). □

Proof of part (iii). Assume that APP is UWT. Note that

$$\sum_{j=1}^{\infty}\frac{\lambda_{d,j}}{\lambda_{d,1}}=\left(\sum_{j=1}^{\infty}\frac{\lambda_{r_d}(j)}{\lambda_{r_d}(1)}\right)^d=\left(\sum_{j=1}^{\infty}\frac{\lambda(j,d)}{\lambda(1,d)}\right)^d\ge(1+h_d)^d.$$

Thus, from Lemma 5 in [10] and the above inequality, we have

$$n^{\mathrm{nor}}(\varepsilon,\mathrm{APP}_d)\ge(1-\varepsilon^2)\left(\sum_{j=1}^{\infty}\frac{\lambda_{d,j}}{\lambda_{d,1}}\right)\ge(1-\varepsilon^2)(1+h_d)^d. \tag{34}$$

Taking the logarithm of the above inequality yields

$$\begin{aligned}\ln n^{\mathrm{nor}}(\varepsilon,\mathrm{APP}_d) &\ge d\ln(1+h_d)+\ln(1-\varepsilon^2) \\ &\ge \frac{dh_d}{2}+\ln(1-\varepsilon^2) \\ &= \frac{d}{2}3^{-(2r_d+2)}+\ln(1-\varepsilon^2),\end{aligned}$$

where, in the second inequality, we use the fact that $\ln(1+x) \geq \frac{1}{2}x$ for $x \in [0,1]$. Due to APP being UWT and fixed $\varepsilon = 1/2$, by (1), we obtain

$$\lim_{d\to\infty} \frac{\ln n^{\text{nor}}(\frac{1}{2}, \text{APP}_d)}{(\frac{1}{2})^{\alpha} + d^{\alpha}} = \lim_{d\to\infty} \frac{\frac{d}{2}3^{-(2r_d+2)} + \ln\frac{3}{4}}{(\frac{1}{2})^{\alpha} + d^{\alpha}} = \lim_{d\to\infty} d^{1-\alpha}3^{-2r_d}$$
$$= \lim_{d\to\infty} d^{1-\alpha-\frac{r_d}{\ln d}2\ln 3} = 0, \quad \text{for all } \alpha > 0.$$

The above equations mean that, for all $\alpha > 0$, we have

$$\frac{r_d}{\ln d} 2\ln 3 > 1 - \alpha$$

for sufficiently large d. It follows that

$$\liminf_{k\to\infty} \frac{r_k}{\ln k} \geq \frac{1}{2\ln 3},$$

as claimed.

On the contrary, assume that $A \geq 1/(2\ln 3)$. This implies that, for every $\delta > 0$, there exists positive integer N_δ such that, for all $k > N_\delta$, we have

$$\frac{r_k}{\ln k} \geq \frac{1-\delta}{2\ln 3},$$

i.e.,

$$r_k \geq \frac{\ln k}{2\ln 3}(1-\delta).$$

By (23), we know that, for all $\alpha \in (0,\infty)$ and all $\tau \in (1/2,1)$, one has

$$\begin{aligned} \frac{d}{d^{\alpha}} \sum_{j=2}^{\infty} \left(\frac{\lambda(j,d)}{\lambda(1,d)}\right)^{\tau} &= d^{1-\alpha} \sum_{j=2}^{\infty} (2j-1)^{-2(r_d+2)\tau} \\ &\leq \frac{2\tau(r_1+1)+2}{2\tau(r_1+1)-1} d^{1-\alpha}3^{-2\tau r_d}. \end{aligned} \tag{35}$$

Now, we use the similar method in [11]. For a fixed $\alpha \in (0,1)$, we set $\delta := \alpha^2/2$ and $\tau := 1 - \alpha^2/2$. Therefore, it is easy to see that $\delta > 0$ and $\tau \in (1/2,1)$. Observe that, for $d > N_\delta$, one has

$$d^{1-\alpha}3^{-2\tau r_d} \leq d^{1-\alpha}3^{-\frac{\ln d}{\ln 3}(1-\delta)\tau} = d^{\alpha^2-\alpha^4/4-\alpha}.$$

On the one hand, by (35), $\alpha^2 - \alpha^4/4 - \alpha < 0$ for $\alpha \in (0,1)$ and the above equation, we obtain

$$\lim_{d\to\infty} \frac{d}{d^{\alpha}} \sum_{j=2}^{\infty} \left(\frac{\lambda(j,d)}{\lambda(1,d)}\right)^{\tau} = \lim_{d\to\infty} d^{\alpha^2-\alpha^4/4-\alpha} = 0. \tag{36}$$

On the other hand, by the proof of Theorem 8 in [10], we see that

$$n^{\text{nor}}(\varepsilon, \text{APP}_d) \leq \left[\exp\left(d \sum_{j=2}^{\infty} \left(\frac{\lambda(j,d)}{\lambda(1,d)}\right)^{\tau}\right) \varepsilon^{-2}\right]^{(1-\tau)^{-1}}, \qquad \forall\, \tau \in (1/2,1). \tag{37}$$

Then, for all $\alpha > 0$, (37) yields

$$\frac{\ln n^{\text{nor}}(\varepsilon, \text{APP}_d)}{\varepsilon^{-\alpha} + d^{\alpha}} \leq \frac{1}{(1-\tau)} \frac{d \sum_{j=2}^{\infty} \left(\frac{\lambda(j,d)}{\lambda(1,d)}\right)^{\tau}}{\varepsilon^{-\alpha} + d^{\alpha}} + \frac{1}{(1-\tau)} \frac{\ln \varepsilon^{-2}}{\varepsilon^{-\alpha} + d^{\alpha}}, \qquad \forall\, \tau \in (1/2,1). \tag{38}$$

Therefore, by combining (36) and (38) , we finally find that (1) holds; i.e., APP is UWT. $\square$

Proof of part (iv). We now turn to (s,t)-WT with $s>0$ and $t>1$. Instead, by replacing α with t in (35) and then combining $3^{-2\tau r_d}\le 1$ and $t>1$, by (35), one obtains

$$\lim_{d\to\infty}\frac{d}{d^t}\sum_{j=2}^{\infty}\left(\frac{\lambda(j,d)}{\lambda(1,d)}\right)^{\tau}=0. \tag{39}$$

Furthermore, from (37), we find that, for all $s,\,t>0$

$$\frac{\ln n^{\rm nor}(\varepsilon,{\rm APP}_d)}{\varepsilon^{-s}+d^t}\le\frac{1}{(1-\tau)}\frac{d\sum_{j=2}^{\infty}\left(\frac{\lambda(j,d)}{\lambda(1,d)}\right)^{\tau}}{\varepsilon^{-s}+d^t}+\frac{1}{(1-\tau)}\frac{\ln\varepsilon^{-2}}{\varepsilon^{-s}+d^t},\qquad\forall\,\tau\in(1/2,1). \tag{40}$$

Thus, for $s>0$ and $t>1$, from (39) and (40), we know that (2) holds; i.e., APP is (s,t)-WT with $s>0$ and $t>1$. $\square$

Proof of part (v). We consider $(s,1)$-WT with $s>0$. Now, we first assume that (17) holds. Then, by letting $\alpha=1$ in (35) first and then using $\lim_{k\to\infty}r_k=\infty$, we find that, for any $\tau\in(1/2,1)$, one has

$$\lim_{d\to\infty}\sum_{j=2}^{\infty}\left(\frac{\lambda(j,d)}{\lambda(1,d)}\right)^{\tau}=\lim_{d\to\infty}3^{-2\tau r_d}=0,\qquad\forall\,\tau\in(1/2,1). \tag{41}$$

Thus, from (40) with $s>0$, $t=1$, and (41), we find that (2) holds; i.e., APP is $(s,1)$-WT with $s>0$.

Conversely, suppose that APP is $(s,1)$-WT with $s>0$. Notice that, if $\lim_{k\to\infty}r_k=r<\infty$, then by (6), we know that $r_d=r$ for sufficiently large d. Therefore, from (34), we obtain

$$n^{\rm nor}(\varepsilon,{\rm APP}_d)\ge(1-\varepsilon^2)(1+h_d)^d\ge(1-\varepsilon^2)\Big(1+3^{-(2r+2)}\Big)^d.$$

The above inequality contradicts $(s,1)$-WT with $s>0$, and, thus, (17) holds. Obviously, the same proof works for WT, which is just the case of $(s,1)$-WT with $s=1$. The proof of (v) is complete. $\square$

Proof of part (vi). We consider (s,t)-WT with $s>0$ and $t\in(0,1)$. Firstly, by using a method similar to the proof of Lemma 2.4 in [9], we can easily find that

$$n^{\rm nor}(\varepsilon,{\rm APP}_d)\ge(1-\varepsilon^2)^{\frac{x+1}{x}}\left(\frac{1+h_d}{1+h_d^{x+1}}\right)^{\frac{d}{x}}, \tag{42}$$

for $x>0$. In the following, we set

$$u_d:=\max(h_d,\frac{1}{2d})\ \text{ and }\ s_d:=\frac{1}{4}(\ln^+\frac{1}{u_d})^{-1}. \tag{43}$$

Now, we consider the necessity. The assumption (s,t)-WT with $s>0$ and $t\in(0,1)$ holds, which implies that $(s,1)$-WT holds with $s>0$; therefore, we have (17). By applying (42) with $x=s_d>0$ and $\varepsilon=\varepsilon_d=(1-\exp(-(s_d+1)^{-1}))^{1/2}$, we obtain

$$\begin{aligned}\ln n^{\rm nor}(\varepsilon_d,{\rm APP}_d)&\ge-\frac{1}{s_d}+\frac{d}{s_d}\ln\left(\frac{1+h_d}{1+h_d^{s_d+1}}\right)\\&\ge-\frac{1}{s_d}+\frac{d}{s_d}\left(\frac{h_d-h_d^{s_d+1}}{1+h_d^{s_d+1}}\right)\ln 2\\&\ge-\frac{1}{s_d}+\frac{d\ln 2}{2s_d}(h_d-h_d^{s_d+1}),\end{aligned} \tag{44}$$

where, in the second inequality, we use the conclusion $\ln(1+x) \geq x \ln 2$ for $0 \leq x \leq 1$. From (43), we have

$$\frac{1}{s_d} = 4\ln^+(\frac{1}{u_d}) \leq 4\ln^+(2d) \text{ and } \lim_{d\to\infty} \frac{1}{s_d d^t} = \lim_{d\to\infty} \frac{4\ln^+(2d)}{d^t} = 0. \tag{45}$$

Since APP is (s,t)-WT, by (44) and (45), we find that

$$\begin{aligned} 0 = \lim_{d\to\infty} \frac{\ln n^{\mathrm{nor}}(\varepsilon_d, \mathrm{APP}_d)}{\varepsilon_d^{-s} + d^t} &\geq \lim_{d\to\infty} \left(-\frac{1}{s_d d^t} + \frac{d\ln 2}{2 s_d d^t}(h_d - h_d^{s_d+1}) \right) \\ &= \frac{\ln 2}{2} \lim_{d\to\infty} \frac{d^{1-t}}{s_d}(h_d - h_d^{s_d+1}) \geq 0, \end{aligned}$$

which implies that

$$\lim_{d\to\infty} \frac{d^{1-t}}{s_d}(h_d - h_d^{s_d+1}) = 0.$$

Then, by continuing with the same technique as in the proof of Theorem 2.1 in [9], we have

$$\lim_{d\to\infty} d^{1-t} h_d \ln \frac{1}{h_d} = 0.$$

Note that $h_d = 1/3^{2r_d+2}$; hence, (18) holds.

We turn to sufficiency. $\sum_{j=1}^{\infty} \overline{\lambda}_{d,j} > 1$; thus, (4) and (5) imply that $n^{\mathrm{nor}}(\varepsilon, \overline{\mathrm{APP}}_d) \leq n^{\mathrm{abs}}(\varepsilon, \overline{\mathrm{APP}}_d)$. Furthermore, from Remark 1, we know that $n^{\mathrm{nor}}(\varepsilon, \mathrm{APP}_d) = n^{\mathrm{nor}}(\varepsilon, \overline{\mathrm{APP}}_d)$. Hence, for $s > 0$ and $t \in (0,1)$, in order to show that APP is (s,t)-WT for the normalized error criterion, it suffices to prove that $\overline{\mathrm{APP}}$ is (s,t)-WT for the absolute error criterion. Note that, for $x \in [3/4, 1)$, one has

$$H(d,x) := \sup_{d\in\mathbb{N}} \sum_{j=2}^{\infty} \left(\frac{\lambda(j,d)}{\lambda(2,d)} \right)^x = \sup_{d\in\mathbb{N}} \sum_{j=1}^{\infty} \left(\frac{3}{2j+1} \right)^{x(2r_d+2)} \leq \sum_{j=1}^{\infty} \left(\frac{3}{2j+1} \right)^{3/2} =: M.$$

Thus, for any $x \in [3/4, 1]$, $d \in \mathbb{N}$,

$$\begin{aligned} \ln\left(\sum_{j=1}^{\infty} \overline{\lambda}_{d,j}^x \right) &= \ln\left(1 + \sum_{j=2}^{\infty} (\lambda(j,d))^x \right)^d \\ &\leq d\ln(1 + h_d^x H(d,x)) \\ &\leq d\ln(1 + h_d^x M) \\ &\leq M d h_d^x, \end{aligned} \tag{46}$$

where we use $\ln(1+x) \leq x$ for $x > 0$ in the last inequality.

From the proof of Theorem 2.1 in [9], we know that, for $\tau \in (0,1)$,

$$n^{\mathrm{abs}}(\varepsilon, \overline{\mathrm{APP}}_d) \leq \varepsilon^{\frac{-2(1-\tau)}{\tau}} \left(\sum_{j=1}^{\infty} \overline{\lambda}_{d,j}^{1-\tau} \right)^{\frac{1}{\tau}}. \tag{47}$$

Now, let $\tau = s_d$, where s_d and u_d are given in (43). Note that, by combining (46) and (47), we have

$$\begin{aligned}\ln n^{\mathrm{abs}}(\varepsilon, \overline{\mathrm{APP}}_d) &\le \frac{2(1-s_d)}{s_d} \ln \varepsilon^{-1} + \frac{1}{s_d} \ln \left(\sum_{j=1}^{\infty} \overline{\lambda}_{d,j}^{1-s_d} \right) \\ &\le \frac{2}{s_d} \ln \varepsilon^{-1} + \frac{Mdh_d^{1-s_d}}{s_d} \\ &\le \frac{2}{s_d} \ln \varepsilon^{-1} + \frac{Mdu_d^{1-s_d}}{s_d}.\end{aligned}$$

Due to

$$u_d^{-s_d} = \exp\left(\frac{\ln \frac{1}{u_d}}{4 \ln^+ \frac{1}{u_d}} \right) \le \mathrm{e}^{1/4},$$

therefore,

$$\ln n^{\mathrm{abs}}(\varepsilon, \overline{\mathrm{APP}}_d) \le \frac{2}{s_d} \ln \varepsilon^{-1} + \frac{\mathrm{e}^{1/4} Mdu_d}{s_d}. \tag{48}$$

Note that

$$\lim_{d\to\infty} d^{1-t} \frac{1}{2d} \ln^+(2d) = 0,$$

thus, by the above equality and (18), we have

$$\lim_{d\to\infty} d^{1-t} u_d \ln^+ \frac{1}{u_d} = 0. \tag{49}$$

On the one hand, by using (45), we have

$$0 \le \lim_{\varepsilon^{-1}+d\to\infty} \frac{\frac{2}{s_d} \ln \varepsilon^{-1}}{\varepsilon^{-s} + d^t} \le \lim_{\varepsilon^{-1}+d\to\infty} \frac{s_d^{-2} + (\ln \varepsilon^{-1})^2}{\varepsilon^{-s} + d^t} = 0. \tag{50}$$

On the other hand, from (49), one finds that

$$0 \le \lim_{\varepsilon^{-1}+d\to\infty} \frac{\frac{du_d}{s_d}}{\varepsilon^{-s} + d^t} = \lim_{\varepsilon^{-1}+d\to\infty} \frac{4du_d \ln^+ \frac{1}{u_d}}{\varepsilon^{-s} + d^t} \le \lim_{\varepsilon^{-1}+d\to\infty} 4d^{1-t} u_d \ln^+ \frac{1}{u_d} = 0. \tag{51}$$

Hence, for $s > 0$ and $t \in (0,1)$, from (48), (50), and (51) we know that (2) holds; i.e., $\overline{\mathrm{APP}}$ is (s,t)-WT with $s > 0$ and $t \in (0,1)$ for the absolute error criterion. □

Proof of part (vii). It follows from (7) that, for $d = 1$, $\lambda_{1,2} = \lambda_{r_1}(2) = (2/3\pi)^{2r_1+2} > 0$. Hence, through statement (1) in Theorem 3.1, in [5], we know that APP is not $(0,t)$-WT. □

Proof of part (viii). We note that the necessity is completely similar to the proof of necessity in (vi), and we omit the details. In the following, we consider the sufficiency. Similar to the discussion in (vi), it suffices to prove that $\overline{\mathrm{APP}}$ is $(s,0)$-WT for the absolute error criterion. Now, we let $s_d = \max\{3/4, 1 - ((\ln d)/(r_d+1))^{1/2}\}$. Note that, in this case,

$$H(d, s_d) \le \sum_{j=1}^{\infty} \left(\frac{3}{2j+1} \right)^{3/2} = M. \tag{52}$$

By using (52) and the same method in (44) and then from (45), we obtain

$$\ln n^{\mathrm{abs}}(\varepsilon, \overline{\mathrm{APP}}_d) \le \frac{2(1-s_d)}{s_d} \ln \varepsilon^{-1} + \frac{Mdh_d^{1-s_d}}{s_d}. \tag{53}$$

By (19), we know that $\lim_{d\to\infty} s_d = 1$ and, hence,

$$\lim_{\varepsilon^{-1}+d\to\infty} \frac{\frac{2(1-s_d)}{s_d}\ln\varepsilon^{-1}}{\varepsilon^{-s}+1} = 0. \tag{54}$$

However, by $\lim_{d\to\infty} s_d = 1$ and (19), we have

$$\lim_{d\to\infty}\left(2r_d(1-s_d) - \frac{\ln d}{\ln 3}\right) = \lim_{d\to\infty}\left(2\sqrt{r_d\ln d} - \frac{\ln d}{\ln 3}\right) = +\infty. \tag{55}$$

Therefore, from (55), one obtains

$$\begin{aligned} 0 \le \lim_{\varepsilon^{-1}+d\to\infty} \frac{\frac{Mdh_d^{1-s_d}}{s_d}}{\varepsilon^{-s}+1} &\le \lim_{d\to\infty} Md3^{-(2r_d+2)(1-s_d)}s_d^{-1} \\ &= \frac{M}{9}\lim_{d\to\infty} 3^{-(2r_d)(1-s_d)+\frac{\ln d}{\ln 3}} = 0. \end{aligned} \tag{56}$$

Hence, for $s > 0$ and $t = 0$, from (53), (54), and (56), we know that (2) holds; i.e., $\overline{\mathrm{APP}}$ is $(s,0)$-WT with $s > 0$ for the absolute error criterion. This completes our proof. □

4. Tractability of Wiener Integrated Process

Now, we consider the various notions of tractability on the Wiener integrated process and derive matching necessary and sufficient conditions, except for $(s,0)$-WT.

Theorem 2. *Consider the multivariate approximation problem APP for the Wiener integrated process. Then, for the normalized error criterion,*

(i) SPT holds if PT holds if

$$A := \liminf_{k\to\infty} \frac{\ln r_k}{\ln k} > \frac{1}{2}. \tag{57}$$

(ii) QPT holds if

$$\sup_{d\in\mathbb{N}} \frac{d(1+r_d)^{-2}\ln^+(1+r_d)}{\ln^+ d} < \infty. \tag{58}$$

(iii) UWT holds if

$$A := \liminf_{k\to\infty} \frac{\ln r_k}{\ln k} \ge \frac{1}{2}.$$

(iv) (s,t)-WT with $s > 0$ and $t > 1$ always holds.

(v) (s,1)-WT with $s > 0$ holds if WT holds if (17) holds.

(vi) (s,t)-WT holds with $s > 0$ and $t \in (0,1)$ if

$$\lim_{k\to\infty} k^{(1-t)}(1+r_k)^{-2}\ln^+(1+r_k) = 0. \tag{59}$$

(vii) APP is not $(0,t)$-WT with $t > 0$.

(viii) If APP is $(s,0)$-WT with $s > 0$, then (59) holds with $t = 0$. However, if

$$\lim_{k\to\infty} \frac{\ln r_k}{\ln k} = \infty, \tag{60}$$

then APP is $(s,0)$-WT with $s > 0$.

Proof of part (i). Based on the logical relationship between SPT and PT, it is obvious that, in order to prove (i), it is suffice to show

$$\mathrm{PT} \Rightarrow \liminf_{k\to\infty} \frac{\ln r_k}{\ln k} > \frac{1}{2} \Rightarrow \mathrm{SPT} \Rightarrow \mathrm{PT}.$$

Now, we let $\lambda_{d,j} = \lambda^{W}_{d,j}$ for $d, j \in \mathbb{N}$, and, thus, for $\tau \in (0,1)$,

$$a_d := \frac{\left(\sum_{j=1}^{\infty} \lambda_{d,j}^{\tau}\right)^{1/\tau}}{\sum_{j=1}^{\infty} \lambda_{d,j}} = b_d^d, \tag{61}$$

where

$$b_d = \frac{\left(1 + Q_d^{\tau} + \sum_{j=3}^{\infty} (\lambda_{r_d}(j)/\lambda_{r_d}(1))^{\tau}\right)^{1/\tau}}{1 + Q_d + \sum_{j=3}^{\infty} \lambda_{r_d}(j)/\lambda_{r_d}(1)}.$$

Since $\lambda_{r_d}(j) = \Theta(j^{-(2r_d+2)})$ as $j \to \infty$, a_d is finite for all d if $(2r_d + 2)\tau > 1$ for all r_d. Therefore, $r_d \geq r_1$ implies that, in what follows, we need to consider $\tau \in (1/(2r_1 + 2), 1)$.

Suppose that APP is PT; then, by (21), we know that $a_d \leq Cd^q$. It is easy to see that $b_d > 1$, which, in turn, implies that $\lim_{d\to\infty} r_d = \infty$. In fact, otherwise, a_d at least exponentially increases on d. However, from (12), for $\tau = 1$, we see that there exists $M > 0$ such that

$$\sum_{j=2}^{\infty} \lambda_{r_d}(j)/\lambda_{r_d}(1) \leq MQ_d.$$

Thus, by dropping the sums over j in the numerator and above inequality, we have

$$\frac{\left(1 + Q_d^{\tau}\right)^{d/\tau}}{(1 + MQ_d)^{d/\tau}} \leq \frac{\left(1 + Q_d^{\tau}\right)^{d/\tau}}{(1 + MQ_d)^{d}} \leq b_d^d < Cd^q.$$

By taking logarithms to the above inequality and from (13), we have

$$\begin{aligned} \ln C + q \ln d &\geq \frac{d}{\tau} \ln\left(\frac{1 + Q_d^{\tau}}{1 + MQ_d}\right) = \frac{d}{\tau} \ln\left(1 + \frac{Q_d^{\tau} - MQ_d}{1 + MQ_d}\right) \\ &\geq \frac{d \ln 2}{(1 + M)\tau} Q_d^{\tau}\left(1 - MQ_d^{1-\tau}\right) = d\Theta(r_d^{-2\tau}), \quad d \to \infty. \end{aligned}$$

Therefore, the above inequality implies $r_d^{-2\tau} = \mathcal{O}(d^{-1} \ln^{+} d)$, and there exists $\delta > 0$ and $N_1 \in \mathbb{N}$ such that

$$r_d \geq \delta \left(\frac{d}{\ln^{+} d}\right)^{1/2\tau} \quad \text{for all } \ d \geq N_1.$$

We now let $s \in (\frac{1}{2}, \frac{1}{2\tau})$ and then have

$$\liminf_{d\to\infty} \frac{r_d}{d^s} > 0.$$

Furthermore, it is easy to prove that the above inequality is equivalent to (57).

Now, we assume that (57) holds, and this implies that $\liminf_{d\to\infty} r_d/d^s > 0$ for some $s > 1/2$. For $\tau \in (\max(3/5, 1/2s), 1]$, by combining this fact and (13), we can conclude that

$$\begin{aligned} \sup_{d\in\mathbb{N}} a_d &= \sup_{d\in\mathbb{N}} \frac{(1 + \mathcal{O}(r_d^{-2\tau}))^{d/\tau}}{(1 + \mathcal{O}(r_d^{-2}))^{d}} \\ &\leq \sup_{d\in\mathbb{N}} \exp\left(d\mathcal{O}(r_d^{-2\tau})\right) \leq \sup_{d\in\mathbb{N}} \exp\left(d\mathcal{O}(d^{-2s\tau})\right) < \infty. \end{aligned} \tag{62}$$

Thus, it follows (21) and (62) that APP is SPT and obviously PT. □

Proof of part (ii). Due to the similarity between this proof and the Euler case, we only sketch it and need to study (29) and (32) for the Wiener eigenvalues. For condition (29), we take $\delta = 1/2$, $\tau_0 \in (3/5, 1)$ and then choose d_0 such that $1 - 1/(2 \ln d_0) \in [\tau_0, 1]$. Hence, for all such $d \geq d_0$, we obtain $\tau_d := 1 - 1/(2\ln d) \in [\tau_0, 1]$, and then we can use the uniform convergence result given in (12). Therefore, in (29),

$$\frac{\sum_{j=1}^{\infty} \lambda_{d,j}^{1-\delta/\ln d}}{\left(\sum_{j=1}^{\infty} \lambda_{d,j}\right)^{1-\delta/\ln d}} = \frac{\left(1 + Q_d^{1-\frac{1}{2\ln d}} + \sum_{j=3}^{\infty}\left(\frac{\lambda_{r_d}(j)}{\lambda_{r_d}(1)}\right)^{1-\frac{1}{2\ln d}}\right)^d}{\left(1 + Q_d + \sum_{j=3}^{\infty}\frac{\lambda_{r_d}(j)}{\lambda_{r_d}(1)}\right)^{d-\frac{d}{2\ln d}}}$$
$$\leq \frac{\left(1 + Q_d^{1-\frac{1}{2\ln d}}(1+\mathcal{O}(r_d^{-h})\right)^d}{(1+Q_d)^{d-\frac{d}{2\ln d}}}.$$

We first suppose that (58) holds. Then, $\lim_{k\to\infty} r_k = \infty$ and (58) implies that

$$(1+Q_d)^{\frac{d}{2\ln d}} \leq \exp\left(\frac{d}{2\ln d}Q_d\right) \leq \exp\left(C\frac{d}{\ln d}r_d^{-2}\right)$$

is uniformly bounded in d. Furthermore, the factor

$$\frac{\left(1 + Q_d^{1-\frac{1}{2\ln d}}(1+\mathcal{O}(r_d^{-h}))\right)^d}{(1+Q_d)^d}$$

can be analyzed exactly as for the Euler case, and, from [8], we have

$$\frac{1 + Q_d^{1-\frac{1}{2\ln d}}(1+\mathcal{O}(r_d^{-h}))}{1+Q_d} \leq 1 + d^{-3/2} + C(1+r_d)^{-2}\left(\frac{\ln^+(r_d+1)}{\ln d} + \mathcal{O}(r_d^{-h})\right).$$

Note that assumption (58) implies $r_d^{-2} = \mathcal{O}(\frac{\ln d}{d})$; thus,

$$\frac{\left(1 + Q_d^{1-\frac{1}{2\ln d}}(1+\mathcal{O}(r_d^{-h}))\right)^d}{(1+Q_d)^d} \leq \exp\left(d\left(d^{-3/2} + C(1+r_d)^{-2}\left(\frac{\ln^+(r_d+1)}{\ln d} + r_d^{-h}\right)\right)\right)$$

is also uniformly bounded in d. Therefore, condition (29) holds, which implies that APP is QPT.

Suppose now that APP is QPT. We use (32) and its consequence (33), which is equivalent to (58). The proof is complete. □

Proof of part (iii). Firstly, we assume that APP is UWT. As before, by taking the logarithm of (34) but replacing h_d with Q_d, we have

$$\begin{aligned}\ln n^{\mathrm{nor}}(\varepsilon, \mathrm{APP}_d) &\geq d\ln(1+Q_d) + \ln(1-\varepsilon^2)\\ &\geq \frac{d}{2}Q_d + \ln(1-\varepsilon^2)\\ &= \Theta(dr_d^{-2}) + \ln(1-\varepsilon^2).\end{aligned}$$

Hence, for fixed ε, by (1), we have

$$\lim_{d\to\infty} d^{-\alpha} d r_d^{-2} = \lim_{d\to\infty} d^{1-\alpha} r_d^{-2} = \lim_{d\to\infty} d^{1-\alpha-2\frac{\ln r_d}{\ln d}} = 0 \ \text{ for all } \ \alpha > 0.$$

This means that, for all $\alpha > 0$, we have

$$2\frac{\ln r_d}{\ln d} > 1 - \alpha$$

for sufficiently large d. Therefore, the above inequality implies that $A \geq \frac{1}{2}$ as claimed.

Conversely, we assume that $A \geq \frac{1}{2}$. That is, for every $\delta > 0$, there exists a positive integer N_δ such that, for all $k > N_\delta$, we have

$$\frac{\ln r_k}{\ln k} \geq \frac{1}{2} - \delta,$$

i.e.,

$$r_k \geq k^{(\frac{1}{2}-\delta)}. \tag{63}$$

Note that, for all $\alpha \in (0, 4/5)$ and all $\tau \in (3/5, 1)$, we have

$$\sum_{j=2}^{\infty} \left(\frac{\lambda_{r_d}(j)}{\lambda_{r_d}(1)} \right)^{\tau} = Q_d^{\tau} \left(1 + \frac{\sum_{j=3}^{\infty} (\lambda_{r_d}(j))^{\tau}}{(\lambda_{r_d}(2))^{\tau}} \right). \tag{64}$$

Hence, it follows (64), (12), and (13) that, for $\tau \in (3/5, 1)$,

$$\sum_{j=2}^{\infty} \left(\frac{\lambda_{r_d}(j)}{\lambda_{r_d}(1)} \right)^{\tau} = \mathcal{O}\left(r_d^{-2\tau} \right) \text{ as } d \to \infty. \tag{65}$$

We now use the similar method in [11] to fix $\alpha \in (0, 4/5)$ and set $\delta := \alpha/4$ and $\tau := 1 - \alpha/2$. Thus, it is obvious to see that $\delta > 0$ and $\tau \in (3/5, 1)$. By (63), we know that, for $d > N_\delta$,

$$d^{1-\alpha} r_d^{-2\tau} \leq d^{1-\alpha} d^{-\tau+2\delta\tau} = d^{1-\alpha} d^{\alpha-\alpha^2/4-1} = d^{-\alpha^2/4}. \tag{66}$$

Therefore, by (65) and (66), we have

$$\lim_{d\to\infty} \frac{d}{d^{\alpha}} \sum_{j=2}^{\infty} \left(\frac{\lambda_{r_d}(j)}{\lambda_{r_d}(1)} \right)^{\tau} = 0.$$

We finally continue with the same proof method as in (iii) of Theorem 1, and, by (31), we find that (1) holds; i.e., APP is UWT. □

Proof of part (iv). By (12) and $\lambda_0^E(j) = \lambda_0^W(j)$, we obtain $M > 0$ such that

$$\sum_{j=2}^{\infty} \left(\frac{\lambda_{r_d}(j)}{\lambda_{r_d}(1)} \right)^{\tau} < M \tag{67}$$

for all $\tau \in (3/5, 1)$. Therefore, by (40), (67), and $t > 1$, we easily observe that (2) holds, which means that (s, t)-WT holds with $s > 0$ and $t > 1$. □

Proof of part (v). We first assume that (17) holds, and then we have $\lim_{d\to\infty} r_d^{-2\tau} = 0$. Based on this fact, (65), and (40), we directly find that APP is $(s, 1)$-WT with $s > 0$.

We now plan to consider necessity, which is verified by the proof of contradiction. Let $\lim_{k\to\infty} r_k < \infty$. By following the same procedure as in the Euler case, we can demonstrate that $n^{\text{nor}}(\varepsilon, \text{APP}_d)$ is an exponential function of d, which contradicts $(s, 1)$-WT. It is obvious that the same works for WT, and we complete the proof of (v). □

Proof of part (vi). The necessity is completely similar to that of the proof in (vi) of Theorem 1. We only need to replace the value of h_d by Q_d and obtain (59).

On the contrary, now we consider the sufficiency. Suppose that condition (59) holds; then, by proof (vi) of Theorem 1, we know that, in order to prove that (s, t)-WT holds with $s > 0$ and $t \in (0, 1)$, we only need to verify for $x \in [3/4, 1)$ that there is a constant $M > 0$ such that $H(d, x) \leq M < \infty$. In fact, by (12) and $\lambda_0^E(j) = \lambda_0^W(j)$, we find that there exists a $D > 0$ such that

$$\sup_{d\in\mathbb{N}} \sum_{j=3}^{\infty} \left(\frac{\lambda_{r_d}(j)}{\lambda_{r_d}(2)} \right)^{x} < D, \quad \forall\, x \in (3/5, 1].$$

Thus, for $x \in [3/4, 1)$, by the above estimate, one has

$$H(d,x) = \sup_{d\in\mathbb{N}} \sum_{j=2}^{\infty} \left(\frac{\lambda_{r_d}(j)}{\lambda_{r_d}(2)} \right)^x = 1 + \sup_{d\in\mathbb{N}} \sum_{j=3}^{\infty} \left(\frac{\lambda_{r_d}(j)}{\lambda_{r_d}(2)} \right)^x \leq 1 + D < \infty. \tag{68}$$

The proof of (vi) is complete. □

Proof of part (vii). From (8), we know that, for $d = 1$, $\lambda_{1,2} = \lambda_{r_1}(2) > 0$. Therefore, statement (1) in Theorem 3.1 of [5] yields that APP is not $(0,t)$-WT. □

Proof of part (viii). The necessity is completely similar to the proof of necessity in (vi) of Theorem 1; thus, we omit the details.

Now, we suppose that (60) holds. Therefore, we can continue the same way as in $(viii)$ of Theorem 1. Firstly, it suffices to prove that $\overline{\text{APP}}$ is $(s,0)$-WT for the absolute error criterion, and we let $s_d = \max\{3/4, 1 - \ln d/\ln(r_d+2)\}$. Similar to the proof process in (53), by using (68) and the same method in (44) and then from (45), we have

$$\ln n^{\text{abs}}(\varepsilon, \overline{\text{APP}}_d) \leq \frac{2(1-s_d)}{s_d} \ln \varepsilon^{-1} + \frac{MdQ_d^{1-s_d}}{s_d}. \tag{69}$$

Note that, by (60), we have $\lim_{d\to\infty} s_d = 1$, and, thus, (54) holds. However, the result $\lim_{d\to\infty} s_d = 1$ and (60) imply that

$$\lim_{d\to\infty} dr_d^{-2(1-s_d)} = \lim_{d\to\infty} d^{1-\frac{2\ln r_d}{\ln(r_d+2)}} = 0. \tag{70}$$

Therefore, from (70) and (13), one has

$$0 \leq \lim_{\varepsilon^{-1}+d\to\infty} \frac{\frac{MdQ_d^{1-s_d}}{s_d}}{\varepsilon^{-s}+1} \leq \lim_{d\to\infty} MCdr_d^{-2(1-s_d)} s_d^{-1} = 0. \tag{71}$$

Hence, for $s > 0$ and $t = 0$, by (69), (54), and (71), we know that (2) holds, and this also means that $\overline{\text{APP}}$ is (s,0)-WT with $s > 0$ for the absolute error criterion. This completes our proof. □

5. Conclusions

In this paper, we discuss the tractability of the multidimensional approximation problem on integrated Euler and Wiener processes with a special case of the covariance kernel of the Gaussian measure and provide sufficient and necessary conditions for various concepts of tractability in terms of the asymptotic properties of the regularity parameters. It should be emphasized that the previous literature only studied some of the tractability concepts, and, in this article, we present a comprehensive study.

We noticed that most articles focused on investigating the normalized error criterion of this problem and that the case of the absolute error criterion is open. In future work, we aim to further study the tractability of the multivariate approximation problem on these two random processes under the absolute error criterion, and the corresponding results will fill the gaps of this field.

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References

1. Novak, E.; Woźniakowski, H. Tractability of multivariate problems, volume I: Linear Information. In *EMS Tracts in Mathematics*; EMS Press: Helsinki, Finland, 2008; Volume 6.
2. Novak, E.; Woźniakowski, H. Tractability of multivariate problems, volume II: Standard Information for functionals. In *EMS Tracts in Mathematics*; EMS Press: Helsinki, Finland, 2010; Volume 12.
3. Novak, E.; Woźniakowski, H. Tractability of multivariate problems, volume III: Standard Information for Operator. In *EMS Tracts in Mathematics*; EMS Press: Helsinki, Finland, 2012; Volume 18.
4. Woźniakowski, H. Tractability and strong tractability of linear multivariate problems. *J. Complex.* **1994**, *10*, 96–128. [CrossRef]
5. Liu, Y.; Xu, G. (S,T)-weak tractability of multivariate linear problems in the average case setting. *Acta Math. Sci.* **2019**, *39*, 1033–1052. [CrossRef]
6. Siedlecki, P.; Weimar, M. Notes on (s,t)-weak tractability: A refined classification of problems with (sub)exponential information complexity. *J. Approx. Theory* **2015**, *200*, 227–258. [CrossRef]
7. Siedlecki, P. Uniform weak tractability. *J. Complex.* **2013**, *29*, 438–453. [CrossRef]
8. Lifshits, M.A.; Papageorgiou, A.; Woźniakowski, H. Tractability of multi-parametric Euler and Wiener integrated processes. *Probab. Math. Statist.* **2012**, *32*, 131–165.
9. Chen, J.; Wang, H.; Zhang, J. Average case (s,t)-weak tractability of non-homogenous tensor product problems. *J. Complex.* **2018**, *49*, 27–45. [CrossRef]
10. Lifshits, M.A.; Papageorgiou, A.; Woźniakowski, H. Average case tractability of non-homogeneous tensor product problems. *J. Complex.* **2012**, *28*, 539–561. [CrossRef]
11. Siedlecki, P. Uniform weak tractability of multivariate problems with increasing smoothness. *J. Complex.* **2014**, *30*, 716–734. [CrossRef]
12. Siedlecki, P. (s,t)-weak tractability of Euler and Wiener integrated processes. *J. Complex.* **2018**, *45*, 55–66. [CrossRef]
13. Xu, G. Average case tractability of non-homogeneous tensor product problems with the absolute error criterion. *J. Complex.* **2023**, *76*, 101743. [CrossRef]
14. Gao, F.; Hanning, J.; Torcaso, F. Integrated Brownian motions and exact L_2-small balls. *Ann. Probab.* **2003**, *31*, 1320–1337. [CrossRef]