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Best Proximity Point Results for Multi-Valued Mappings in Generalized Metric Structure

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Abstract: In this paper, we introduce the novel concept of generalized distance denoted as J_θ and call it an extended b-generalized pseudo-distance. With the help of this generalized distance, we define a generalized point to set distance $J_\theta(\mathfrak{u}, \mathcal{H}^\star)$, a generalized Hausdorff type distance and a P^{J_θ}-property of a pair $(\mathcal{H}^\star, \mathcal{K}^\star)$ of nonempty subsets of extended b-metric space $(\mathfrak{U}^\star, \rho_\theta)$. Additionally, we establish several best proximity point theorems for multi-valued contraction mappings of Nadler type defined on b-metric spaces and extended b-metric spaces. Our findings generalize numerous existing results found in the literature. To substantiate the introduced notion and validate our main results, we provide some concrete examples.

Keywords: best proximity point; fixed point; multi-valued contraction; b-metric; extended b-generalized pseudo-distance

1. Introduction

Fréchet [1] began the study of spaces with distance functions in 1905 by giving every pair of generic objects in a nonempty set a non-negative value. These spaces were subsequently termed metric spaces by Hausdorff. In these types of spaces, the distance between two objects is specified by a metric function or distance function that, in addition to being non-negativity, also has the triangle inequality, symmetry, and identity of indiscernibles. There are plenty of metric space generalizations, and most of them are accomplished by eradicating, weakening, or expanding one of the aforementioned features. (see, for example, refs. [2–7] and the references therein). One of the generalizations of metric space is symmetric space. The triangular inequality of a metric function is removed in symmetric spaces (see [8,9]), and several substitutes of the triangular inequality are used to demonstrate different features and the existence of a fixed point (abbreviated as F. point) of contractive type mappings. Numerous researchers developed significant F. point theorems in the context of symmetric spaces, which they applied to the split minimization problem, the split feasibility problem, and the positive solutions of fractional periodic boundary value problems (see, for example, [10,11]). A fundamental result in F. point theory is the Banach contraction principle. Several extensions of this result have appeared in the literature (see, for example, refs. [12–14] and the references cited therein). We can also find various generalizations of the variant Banach contraction principle using the graph theoretic approach. The graphs considered by Jachymski [15] are such that $G = (V(G); E(G))$, where $V(G)$ is the set of vertices and $E(G)$ is the set of edges, satisfy the conditions $V(G) = \mathfrak{U}^\star$ and $\Delta_{\mathfrak{U}^\star} \subseteq E(G) \subseteq \mathfrak{U}^\star \times \mathfrak{U}^\star$, so that $E(G)$ is, in fact, a reflexive binary relation on $\mathfrak{U}^\star$. Here, $\Delta_{\mathfrak{U}^\star} = \{(\mathfrak{u}, \mathfrak{u}) : \mathfrak{u} \in \mathfrak{U}^\star\}$ is the diagonal of $\mathfrak{U}^\star \times \mathfrak{U}^\star$. Other recent results for single-valued and multi-valued operators in metric spaces endowed with graphs are given by Bojor [16], Aleomraninejada et al. [17], Beg et al. [18] and by Chifu et al. [19].

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In a metric space $(\mathfrak{U}^\star, \rho)$, the F. point of a multi-valued mapping $\Gamma : \mathfrak{U}^\star \to 2^{\mathfrak{U}^\star}$ is an element $\mathfrak{p} \in \mathfrak{U}^\star$, such that $\mathfrak{p} \in \Gamma\mathfrak{p}$. If $\Gamma\mathfrak{p}$ is a closed subset of $\mathfrak{U}^\star$, then $\mathfrak{p} \in \mathfrak{U}^\star$ is a F. point of Γ if $D(\mathfrak{p}, \Gamma\mathfrak{p}) = 0$, where $D(\mathfrak{p}, \Gamma\mathfrak{p}) = \inf_{\mathfrak{u} \in \Gamma\mathfrak{p}} \rho(\mathfrak{p}, \mathfrak{u})$. Now, if $\mathcal{H}^\star$ and $\mathcal{K}^\star$ are nonempty subsets of a metric space $(\mathfrak{U}^\star, \rho)$ and $\Gamma : \mathcal{H}^\star \to 2^{\mathcal{K}^\star}$ is a multi-valued mapping. Then it is not necessary that Γ has a F. point $\mathfrak{p}$ in $\mathcal{H}^\star$. The idea of best proximity point originates here. A best proximity point (abbreviated as B.P. point) of the multi-valued mapping $\Gamma : \mathcal{H}^\star \to 2^{\mathcal{K}^\star}$ is an element $\mathfrak{p} \in \mathcal{H}^\star$, such that $D(\mathfrak{p}, \Gamma\mathfrak{p}) = dst(\mathcal{H}^\star, \mathcal{K}^\star)$ where $dst(\mathcal{H}^\star, \mathcal{K}^\star) = \inf\{\rho(\mathfrak{u}, \mathfrak{e}) : \mathfrak{u} \in \mathcal{H}^\star, \mathfrak{e} \in \mathcal{K}^\star\}$. If $\Gamma : \mathcal{H}^\star \to \mathcal{K}^\star$ is a non-self single-valued mapping, then an element $\mathfrak{p} \in \mathcal{H}^\star$ is called a B.P. point of Γ if $\rho(\mathfrak{p}, \Gamma\mathfrak{p}) = dst(\mathcal{H}^\star, \mathcal{K}^\star)$. Many researchers have been interested in the topics of B.P. points for single-valued and multi-valued mappings in recent years. For single-valued mappings, the existence of B.P. points was established by S. Sadiq Basha et al. [20], C. Di Baria et al. [21], M.A. Al-Thagafi and N. Shahzad [22], D. Sarkar [23], and many others. B.P. point theorems for multi-valued mappings were established by G. AlNemer et al. [24], K. Włodarczyk and R. Plebaniak [25], A. Abkar and M. Gabeleh [26], M.A. Al-Thagafi and N. Shahzad [27], M. Gabeleh [28], and many others. In 2014, Plebaniak [29] established an important B.P. point theorem by adopting the following definitions.

Definition 1 ([29])**.** *Let $(\mathfrak{U}^\star, \rho_b)$ be a b-metric space (b-m space) (with constant $s \geq 1$). A mapping $J_b : \mathfrak{U}^\star \times \mathfrak{U}^\star \to [0, \infty)$ is said to be a b-generalized pseudo-distance (b-G pseudo-distance) on $\mathfrak{U}^\star$ if it satisfies*

(J_b1) $J_b(\mathfrak{u}, \mathfrak{t}) \leq s[J_b(\mathfrak{u}, \mathfrak{e}) + J_b(\mathfrak{e}, \mathfrak{t})]\ \forall\ \mathfrak{u}, \mathfrak{e}, \mathfrak{t} \in \mathfrak{U}^\star$.

(J_b2) *For any sequences $\{\mathfrak{u}_n : n \in \mathbb{N}\}$ and $\{\mathfrak{e}_n : n \in \mathbb{N}\}$ in $\mathfrak{U}^\star$ satisfying*

$$\lim_{m\to\infty} \sup_{n>m} J_b(\mathfrak{u}_m, \mathfrak{u}_n) = 0, \tag{1}$$

and

$$\lim_{n\to\infty} J_b(\mathfrak{u}_n, \mathfrak{e}_n) = 0, \tag{2}$$

the following holds

$$\lim_{n\to\infty} \rho_b(\mathfrak{u}_n, \mathfrak{e}_n) = 0. \tag{3}$$

Throughout, let the triplet $(\mathfrak{U}^\star, \rho_b, J_b)$ denote a b-m space $(\mathfrak{U}^\star, \rho_b)$ (with $s \geq 1$) equipped with a b-G pseudo-distance J_b.

Definition 2 ([29])**.** *Let $\mathcal{H}^\star$ and $\mathcal{K}^\star$ be the subsets of a topological space $(\mathfrak{U}^\star, \tau)$. A multi-valued mapping $\Gamma : \mathcal{H}^\star \to 2^{\mathcal{K}^\star}$ is called closed whenever a sequence $\{\mathfrak{u}_n : n \in \mathbb{N}\}$ in $\mathcal{H}^\star$ converges to $\mathfrak{u} \in \mathcal{H}^\star$ and a sequence $\{\mathfrak{e}_n : n \in \mathbb{N}\}$ in $\mathcal{K}^\star$ converges to $\mathfrak{e} \in \mathcal{K}^\star$, such that $\mathfrak{e}_n \in \Gamma(\mathfrak{u}_n)$, for all $n \in \mathbb{N}$, implying that $\mathfrak{e} \in \Gamma(\mathfrak{u})$.*

Theorem 1 ([29])**.** *Let $\mathcal{H}^\star$ and $\mathcal{K}^\star$ be the subsets of a complete space $(\mathfrak{U}^\star, \rho_b, J_b)$, such that they are closed, J_b is associated with $(\mathcal{H}^\star, \mathcal{K}^\star)$, and $(\mathcal{H}^\star, \mathcal{K}^\star)$ has the P^{J_b}-property. Let $\Gamma : \mathcal{H}^\star \to 2^{\mathcal{K}^\star}$ be a closed, multi-valued mapping, such that*

$$sH^{J_b}(\Gamma\mathfrak{u}, \Gamma\mathfrak{e}) \leq kJ_b(\mathfrak{u}, \mathfrak{e})\ \forall\ \mathfrak{u}, \mathfrak{e} \in \mathfrak{U}^\star, \tag{4}$$

for some $0 \leq k < 1$. Let $\Gamma\mathfrak{u} \in CB(\mathfrak{U}^\star)$, $\Gamma\mathfrak{t} \subset \mathcal{K}_0$ for each $\mathfrak{u} \in \mathcal{H}^\star$, $\mathfrak{t} \in \mathcal{H}_0$. Then $D(\mathfrak{p}, \Gamma\mathfrak{p}) = dst(\mathcal{H}^\star, \mathcal{K}^\star)$ for some $\mathfrak{p} \in \mathcal{H}^\star$.

2. Preliminaries

The triangle inequality related to the metric function is important in the context of B.P point theorems and fixed-point theorems for demonstrating the convergence of an iterative sequence. Consequently, a number of authors have endeavored to identify spaces where the triangle inequality was incorporated in a more mild or comprehensive manner, guaranteeing that the presence of a fixed point or B.P. point could still be proven. In

1993, Czerwik [3] gave a weaker axiom than the triangular inequality of metric space and formally defined the notion of b-metric space. Afterward, Fagin et al. [30] argued about a relaxation of the triangle inequality and named this new distance measure non-linear elastic math (NEM). A comparable form of the relaxed triangle inequality was also applied to the measurement of ice floes [31] and trade [32]. Because of all those applications, Kamran et al. [13] was able to present the following definition of extended b-metric space.

Definition 3 ([13])**.** *Let $\mathfrak{U}^\star$ be a nonempty set and $\theta : \mathfrak{U}^\star \times \mathfrak{U}^\star \to [1, \infty)$.*

A function $\rho_\theta : \mathfrak{U}^\star \times \mathfrak{U}^\star \to [0, \infty)$ is called an extended b-metric if for each $\mathfrak{u}, \mathfrak{e}, \mathfrak{t} \in \mathfrak{U}^\star$ it satisfies

($M_\theta 1$): $\rho_\theta(\mathfrak{u}, \mathfrak{e}) = 0$ *iff* $\mathfrak{u} = \mathfrak{e}$;

($M_\theta 2$): $\rho_\theta(\mathfrak{u}, \mathfrak{e}) = \rho_\theta(\mathfrak{e}, \mathfrak{u})$;

($M_\theta 3$): $\rho_\theta(\mathfrak{u}, \mathfrak{t}) \leq \theta(\mathfrak{u}, \mathfrak{t})[\rho_\theta(\mathfrak{u}, \mathfrak{e}) + \rho_\theta(\mathfrak{e}, \mathfrak{t})]$.

The pair $(\mathfrak{U}^\star, \rho_\theta)$ is called an extended b-metric space (E.b-m space).

Remark 1. *A b-m space becomes a special case of E.b-m space when $\theta(\mathfrak{u}, \mathfrak{e}) = s$ for $s \geq 1, \mathfrak{u}, \mathfrak{e} \in \mathfrak{U}^\star$.*

Example 1 ([33])**.** *Let $\mathfrak{U}^\star = [0, 1]$. Define $\theta : \mathfrak{U}^\star \times \mathfrak{U}^\star \to [0, \infty)$ as*

$$\theta(\mathfrak{u}, \mathfrak{e}) = \begin{cases} \frac{1+\mathfrak{u}+\mathfrak{e}}{\mathfrak{u}+\mathfrak{e}} & \text{for} \quad \mathfrak{u} \neq \mathfrak{e} \neq 0 \\ 1 & \text{for} \quad \mathfrak{u} = \mathfrak{e} = 0. \end{cases}$$

Let $\rho_\theta : \mathfrak{U}^\star \times \mathfrak{U}^\star \to [0, \infty)$ be defined by

$$\rho_\theta(\mathfrak{u}, \mathfrak{e}) = \frac{1}{\mathfrak{u}\mathfrak{e}} \quad \text{for } \mathfrak{u}, \mathfrak{e} \in (0, 1] \text{ with } \mathfrak{u} \neq \mathfrak{e},$$

$$\rho_\theta(\mathfrak{u}, \mathfrak{e}) = 0 \quad \text{for } \mathfrak{u}, \mathfrak{e} \in [0, 1] \text{ with } \mathfrak{u} = \mathfrak{e},$$

$$\rho_\theta(\mathfrak{u}, 0) = \rho_\theta(0, \mathfrak{u}) = \frac{1}{\mathfrak{u}} \quad \text{for } \mathfrak{u} \in (0, 1].$$

Then $(\mathfrak{U}^\star, \rho_\theta)$ is E.b-m space.

Note. Throughout this manuscript, we assume that the E.b-metric ρ_θ is continuous on $\mathfrak{U}^{\star 2} = \mathfrak{U}^\star \times \mathfrak{U}^\star$.

The following is the main result of [13].

Theorem 2 ([13])**.** *Let $(\mathfrak{U}^\star, \rho_\theta)$ be a complete E.b-m space and a mapping $\Gamma : \mathfrak{U}^\star \to \mathfrak{U}^\star$ satisfy*

$$\rho_\theta(\Gamma\mathfrak{u}, \Gamma\mathfrak{e}) \leq k\rho_\theta(\mathfrak{u}, \mathfrak{e}) \ \forall \ \mathfrak{u}, \mathfrak{e} \ \in \mathfrak{U}^\star,$$

for some $0 \leq k < 1$ such that for each $\mathfrak{u}_0 \in \mathfrak{U}^\star$, $\lim_{n,m\to\infty} \theta(\mathfrak{u}_n, \mathfrak{u}_m) < \frac{1}{k}$, here $\mathfrak{u}_n = \Gamma^n(\mathfrak{u}_0)$, $n = 1, 2, 3, \cdots$. Then $\rho_\theta(\mathfrak{p}, \Gamma\mathfrak{p}) = 0$ for a unique $\mathfrak{p} \in \mathfrak{U}^\star$. Moreover, for each $\mathfrak{e} \in \mathfrak{U}^\star$, $\Gamma^n(\mathfrak{e}) \to \mathfrak{p}$.

Drawing inspiration from the concept of extended b-metric and $\mathfrak{b}$-generalized pseudo-distances, we introduce the novel concept of generalized distance, within an extended b-metric space. This notion extends, generalizes, and improves the notion of E.b-metric and the notion of b-generalized pseudo-distances. Furthermore, some B.P. point theorems are proved in this new framework, which generalizes and extends many previous findings in the literature. In order to clarify and validate ideas and claims, numerous examples are offered.

3. Main Results

In the following, we start by formulating our notion.

Definition 4. *Let $(\mathfrak{U}^\star, \rho_\theta)$ be an E.b-m space. A mapping $J_\theta : \mathfrak{U}^\star \times \mathfrak{U}^\star \to [0, \infty)$ is said to be an extended b-generalized pseudo-distance (E.b-G pseudo-distance) on $\mathfrak{U}^\star$ if it satisfies*
$(J_\theta 1)$ $J_\theta(\mathfrak{u}, \mathfrak{t}) \leq \theta(\mathfrak{u}, \mathfrak{t})[J_\theta(\mathfrak{u}, \mathfrak{e}) + J_\theta(\mathfrak{e}, \mathfrak{t})]\ \forall\ \mathfrak{u}, \mathfrak{e}, \mathfrak{t} \in \mathfrak{U}^\star$.
$(J_\theta 2)$ *For any sequences $\{\mathfrak{u}_n : n \in \mathbb{N}\}$ and $\{\mathfrak{e}_n : n \in \mathbb{N}\}$ in $\mathfrak{U}^\star$ satisfying*

$$\lim_{m\to\infty} \sup_{n>m} J_\theta(\mathfrak{u}_m, \mathfrak{u}_n) = 0, \tag{5}$$

and

$$\lim_{n\to\infty} J_\theta(\mathfrak{u}_n, \mathfrak{e}_n) = 0, \tag{6}$$

the following holds

$$\lim_{n\to\infty} \rho_\theta(\mathfrak{u}_n, \mathfrak{e}_n) = 0. \tag{7}$$

Remark 2. *Every E.b-metric $\rho_\theta : \mathfrak{U}^\star \times \mathfrak{U}^\star \to [0, \infty)$ on $\mathfrak{U}^\star$ is an E.b-G pseudo-distance on $\mathfrak{U}^\star$, but the converse is not true in general.*

Example 2. *Let $\mathfrak{B}^\star$ be a closed subset of $(\mathfrak{U}^\star, \rho_\theta)$ such that it contains at least two points. Let $r > 0$ with $r > \delta(\mathfrak{B}^\star)$ where $\delta(\mathfrak{B}^\star) = \sup\{\rho_\theta(\mathfrak{u}, \mathfrak{e}); \mathfrak{u}, \mathfrak{e} \in \mathfrak{B}^\star\}$.*
Define $J_\theta : \mathfrak{U}^\star \times \mathfrak{U}^\star \to [0, \infty)$ as

$$J_\theta(\mathfrak{u}, \mathfrak{e}) = \begin{cases} \rho_\theta(\mathfrak{u}, \mathfrak{e}) & \text{if } \{\mathfrak{u}, \mathfrak{e}\} \subseteq \mathfrak{B}^\star \\ r & \text{if } \{\mathfrak{u}, \mathfrak{e}\} \nsubseteq \mathfrak{B}^\star. \end{cases}$$

Then, J_θ is an extended b-generalized pseudo-distance on $\mathfrak{U}^\star$.

Proof. $(J_\theta 1)$ Let $\mathfrak{u}_0, \mathfrak{e}_0, \mathfrak{t}_0 \in \mathfrak{U}^\star$ satisfy

$$J_\theta(\mathfrak{u}_0, \mathfrak{t}_0) > \theta(\mathfrak{u}_0, \mathfrak{t}_0)[J_\theta(\mathfrak{u}_0, \mathfrak{e}_0) + J_\theta(\mathfrak{e}_0, \mathfrak{t}_0)]. \tag{8}$$

Then, $\{\mathfrak{u}_0, \mathfrak{e}_0, \mathfrak{t}_0\}$ is not a subset of $\mathfrak{B}^\star$, because if it is a subset of $\mathfrak{B}^\star$, then

$$J_\theta(\mathfrak{u}_0, \mathfrak{e}_0) = \rho_\theta(\mathfrak{u}_0, \mathfrak{e}_0), J_\theta(\mathfrak{u}_0, \mathfrak{t}_0) = \rho_\theta(\mathfrak{u}_0, \mathfrak{t}_0), J_\theta(\mathfrak{e}_0, \mathfrak{t}_0) = \rho_\theta(\mathfrak{e}_0, \mathfrak{t}_0).$$

So (8) becomes

$$\rho_\theta(\mathfrak{u}_0, \mathfrak{t}_0) > \theta(\mathfrak{u}_0, \mathfrak{t}_0)[\rho_\theta(\mathfrak{u}_0, \mathfrak{e}_0) + \rho_\theta(\mathfrak{e}_0, \mathfrak{t}_0)].$$

This is a contradiction to the fact that ρ_θ is an extended b-metric on $\mathfrak{U}^\star$. Therefore, there exists some $\mathfrak{u} \in \{\mathfrak{u}_0, \mathfrak{e}_0, \mathfrak{t}_0\}$, such that $\mathfrak{u} \notin \mathfrak{B}^\star$. If $\mathfrak{u} = \mathfrak{u}_0$, then $J_\theta(\mathfrak{u}_0, \mathfrak{t}_0) = r$, and $J_\theta(\mathfrak{u}_0, \mathfrak{e}_0) = r$. Thus (8) becomes $r > \theta(\mathfrak{u}_0, \mathfrak{t}_0)[r + J_\theta(\mathfrak{e}_0, \mathfrak{t}_0)]$, which is a contradiction. Similarly, if we take $\mathfrak{u} = \mathfrak{e}_0$ or $\mathfrak{u} = \mathfrak{t}_0$, then we obtain a contradiction. Hence, the condition $(J_\theta 1)$ is fulfilled, i.e.,

$$J_\theta(\mathfrak{u}, \mathfrak{t}) \leq \theta(\mathfrak{u}, \mathfrak{t})[J_\theta(\mathfrak{u}, \mathfrak{e}) + J_\theta(\mathfrak{e}, \mathfrak{t})] \text{ for all } \mathfrak{u}, \mathfrak{e}, \mathfrak{t} \in \mathfrak{U}^\star.$$

$(J_\theta 2)$ Let $\{\mathfrak{u}_n : n \in \mathbb{N}\}$ and $\{\mathfrak{e}_n : n \in \mathbb{N}\}$ be any sequences in $\mathfrak{U}^\star$ such that $\lim_{m\to\infty} \sup_{n>m} J_\theta(\mathfrak{u}_m, \mathfrak{u}_n) = 0$ and $\lim_{n\to\infty} J_\theta(\mathfrak{u}_n, \mathfrak{e}_n) = 0$. We show that

$$\lim_{n\to\infty} \rho_\theta(\mathfrak{u}_n, \mathfrak{e}_n) = 0.$$

Since $\lim_{n\to\infty} J_\theta(\mathfrak{u}_n, \mathfrak{e}_n) = 0$, $\lim_{n\to\infty} \mathfrak{t}_n = 0$ where $\mathfrak{t}_n = J_\theta(\mathfrak{u}_n, \mathfrak{e}_n) \in R^+$, which further implies that for $0 < \varepsilon < r$ there is some $k \in \mathbb{N}$, such that

$$\mathfrak{t}_n < \varepsilon < r \text{ whenever } n \geq k.$$

From this we obtain the following:

$$J_\theta(\mathfrak{u}_n, \mathfrak{e}_n) < \varepsilon < r \text{ whenever } n \geq k,$$

$$\rho_\theta(\mathfrak{u}_n, \mathfrak{e}_n) < \varepsilon \text{ whenever } n \geq k.$$

Since $\mathfrak{t}_n = J_\theta(\mathfrak{u}_n, \mathfrak{e}_n)\ \forall\ n \in \mathbb{N}$, and $J_\theta(\mathfrak{u}, \mathfrak{e}) = \rho_\theta(\mathfrak{u}, \mathfrak{e})\ \forall\ \mathfrak{u}, \mathfrak{e} \in \mathfrak{U}^\star$, when $J_\theta(\mathfrak{u}, \mathfrak{e}) \neq r$, $\lim_{n\to\infty} \rho_\theta(\mathfrak{u}_n, \mathfrak{e}_n) = 0$. □

Note. Throughout, let the triplet $(\mathfrak{U}^\star, \rho_\theta, J_\theta)$ denote an E.b-m space $(\mathfrak{U}^\star, \rho_\theta)$ equipped with the E.b-G pseudo-distance J_θ.

Remark 3. *An E.b-G pseudo-distance need not be a b-G pseudo-distance.*

The following counter-example validates Remark 3.

Example 3. *Let $\mathfrak{U}^\star = [0,1]$. Define $\theta : \mathfrak{U}^\star \times \mathfrak{U}^\star \to [0,\infty)$ as*

$$\theta(\mathfrak{u}, \mathfrak{e}) = \begin{cases} \frac{1+\mathfrak{u}\mathfrak{e}}{\mathfrak{u}+\mathfrak{e}} & \text{for} \quad \mathfrak{u}, \mathfrak{e} \in [0,1] \\ \frac{3}{2} & \text{for} \quad \mathfrak{u} = \mathfrak{e} = 0. \end{cases}$$

Let $\rho_\theta : \mathfrak{U}^\star \times \mathfrak{U}^\star \to [0,\infty)$ be defined by

$$\rho_\theta(\mathfrak{u}, \mathfrak{e}) = \frac{1}{\mathfrak{u}\mathfrak{e}} \quad \text{if } \mathfrak{u}, \mathfrak{e} \in (0,1] \text{ and } \mathfrak{u} \neq \mathfrak{e},$$

$$\rho_\theta(\mathfrak{u}, \mathfrak{e}) = 0 \quad \text{if } \mathfrak{u}, \mathfrak{e} \in [0,1] \text{ and } \mathfrak{u} = \mathfrak{e},$$

$$\rho_\theta(\mathfrak{u}, 0) = \rho_\theta(0, \mathfrak{u}) = \frac{1}{\mathfrak{u}} \quad \text{if } \mathfrak{u} \in (0,1].$$

Then, $(\mathfrak{U}^\star, \rho_\theta)$ is an E.b-m space. (See [34]). Let $J_\theta : \mathfrak{U}^\star \times \mathfrak{U}^\star \to [0,\infty)$ be defined by

$$J_\theta(\mathfrak{u}, \mathfrak{e}) = \rho_\theta(\mathfrak{u}, \mathfrak{e}) \text{ for all } \mathfrak{u}, \mathfrak{e} \in \mathfrak{U}^\star.$$

Then, by Remark 2, the map J_θ is an E.b-G pseudo-distance on $\mathfrak{U}^\star$. We show that J_θ is not a b-G pseudo-distance on $\mathfrak{U}^\star$. Let us suppose, on the contrary, J_θ be a b-G pseudo-distance on $\mathfrak{U}^\star$. Then, there is some $s \geq 1$, such that

$$J_\theta(\mathfrak{u}, \mathfrak{t}) \leq s[J_\theta(\mathfrak{u}, \mathfrak{e}) + J_\theta(\mathfrak{e}, \mathfrak{t})] \text{ for all } \mathfrak{u}, \mathfrak{e}, \mathfrak{t} \in \mathfrak{U}^\star. \tag{9}$$

Now, if $\mathfrak{l} > 1$, then $\mathfrak{l}+1 > 1$ and $\frac{1}{\mathfrak{l}}, \frac{1}{\mathfrak{l}+1} \in (0,1]$. Let $\mathfrak{u} = \frac{1}{\mathfrak{l}}$, $\mathfrak{e} = 0$, $\mathfrak{t} = \frac{1}{\mathfrak{l}+1}$, then (9) becomes

$$J_\theta\left(\frac{1}{\mathfrak{l}}, \frac{1}{\mathfrak{l}+1}\right) \leq s\left[J_\theta\left(\frac{1}{\mathfrak{l}}, 0\right) + J_\theta\left(0, \frac{1}{\mathfrak{l}+1}\right)\right].$$

So $\mathfrak{l}(\mathfrak{l}+1) \leq s[\mathfrak{l}+\mathfrak{l}+1]$. Thus

$$\mathfrak{l}^2 + \mathfrak{l} \leq s[2\mathfrak{l}+1]. \tag{10}$$

Let $\mathfrak{l} = 3s+3 > 1$. Then (10) becomes

$$(3s+3)^2 + 3s + 3 \leq s[2(3s+3)+1]$$

and so

$$3s^2 + 14s + 12 \leq 0,$$

which is a contradiction to $s \geq 1$. Thus, J_θ is not a b-G pseudo-distance on $\mathfrak{U}^\star$.

We formulate the following definitions.

Definition 5. *Let $\mathcal{H}^\star$ and $\mathcal{K}^\star$ be nonempty subsets of $(\mathfrak{U}^\star, \rho_\theta, J_\theta)$. Define*

$$J_\theta(\mathfrak{u}, \mathcal{K}^\star) = \inf_{\mathfrak{e} \in \mathcal{K}^\star} J_\theta(\mathfrak{u}, \mathfrak{e}),$$

$$dst(\mathcal{H}^\star, \mathcal{K}^\star) = \inf\{J_\theta(\mathfrak{u}, \mathfrak{e}) : \mathfrak{u} \in \mathcal{H}^\star,\ \mathfrak{e} \in \mathcal{K}^\star\},$$

$$\mathcal{H}_0 = \{\mathfrak{u}\ \in \mathcal{H}^\star : J_\theta(\mathfrak{u}, \mathfrak{e}) = dst(\mathcal{H}^\star, \mathcal{K}^\star) \text{ for some } \mathfrak{e}\ \in \mathcal{K}^\star\},$$

$$\mathcal{K}_0 = \{\mathfrak{e}\ \in \mathcal{K}^\star : J_\theta(\mathfrak{u}, \mathfrak{e}) = dst(\mathcal{H}^\star, \mathcal{K}^\star) \text{ for some } \mathfrak{u} \in \mathcal{H}^\star\}.$$

$$H^{J_\theta} : CB(\mathfrak{U}^\star) \times CB(\mathfrak{U}^\star) \to [0, \infty) \text{ by}$$

$$H^{J_\theta}(\mathcal{H}^\star, \mathcal{K}^\star) = \max\{\sup_{\mathfrak{u} \in \mathcal{H}^\star} J_\theta(\mathfrak{u}, \mathcal{K}^\star), \sup_{\mathfrak{e} \in \mathcal{K}^\star} J_\theta(\mathfrak{e}, \mathcal{H}^\star)\} \text{ for all } \mathcal{H}^\star, \mathcal{K}^\star\ \in CB(\mathfrak{U}^\star).$$

Here, $CB(\mathfrak{U}^\star) = \{S \subset \mathfrak{U}^\star;\ S$ is closed and bounded$\}$.

Definition 6. *Let $\mathcal{H}^\star$ and $\mathcal{K}^\star$ be the subsets of $(\mathfrak{U}^\star, \rho_\theta, J_\theta)$ with $\mathcal{H}_0 \neq \emptyset$. Then:*
(i) The pair $(\mathcal{H}^\star, \mathcal{K}^\star)$ is said to have a P^{J_θ}-property if and only if

$$\begin{cases} J_\theta(\mathfrak{u}_1, \mathfrak{e}_1) = dst(\mathcal{H}^\star, \mathcal{K}^\star), \\ J_\theta(\mathfrak{u}_2, \mathfrak{e}_2) = dst(\mathcal{H}^\star, \mathcal{K}^\star) \end{cases} \implies \quad J_\theta(\mathfrak{u}_1, \mathfrak{u}_2) = J_\theta(\mathfrak{e}_1, \mathfrak{e}_2),$$

where $\mathfrak{u}_1, \mathfrak{u}_2 \in \mathcal{H}_0$, and $\mathfrak{e}_1, \mathfrak{e}_2 \in \mathcal{K}_0$.
(ii) An E.b-G pseudo-distance J_θ is said to be associated with $(\mathcal{H}^\star, \mathcal{K}^\star)$, if for any sequences $\{\mathfrak{u}_n : n \in \mathbb{N}\}$ and $\{\mathfrak{e}_n : n \in \mathbb{N}\}$ in $\mathfrak{U}^\star$, such that

$$\lim_{n\to\infty} \mathfrak{u}_n = \mathfrak{u};\ \lim_{n\to\infty} \mathfrak{e}_n = \mathfrak{e};\ \text{and } J_\theta(\mathfrak{u}_n, \mathfrak{e}_{n-1}) = dst(\mathcal{H}^\star, \mathcal{K}^\star)\ \forall\ n \in \mathbb{N},$$

we have $\rho_\theta(\mathfrak{u}, \mathfrak{e}) = dst(\mathcal{H}^\star, \mathcal{K}^\star)$.

It is clear that for E.b-metric space $(\mathfrak{U}^\star, \rho_\theta)$ if we put $J_\theta = \rho_\theta$, then the mapping ρ_θ is associated with each pair $(\mathcal{H}^\star, \mathcal{K}^\star)$ of nonempty subsets of $\mathfrak{U}^\star$, because of the continuity of ρ_θ (we have chosen ρ_θ to be continuous throughout).

The following lemmas are important to prove our main result.

Lemma 1. *Let $\{\mathfrak{u}_n : n \in \{0\} \cup \mathbb{N}\}$ be a sequence in the complete space $(\mathfrak{U}^\star, \rho_\theta, J_\theta)$, such that*

$$\lim_{m\to\infty} \sup_{n>m} J_\theta(\mathfrak{u}_m, \mathfrak{u}_n) = 0, \tag{11}$$

and $\lim_{n,m\to\infty} \theta(\mathfrak{u}_n, \mathfrak{u}_m)$ exists and is finite. Then $\{\mathfrak{u}_n : n \in \{0\} \cup \mathbb{N}\}$ is Cauchy.

Proof. From (11) we can write that

$$\forall\ \varepsilon > 0,\ \exists\ m_1 = m_1(\varepsilon) \in \mathbb{N},\ \forall\ m > m_1,\ \{\sup\{J_\theta(\mathfrak{u}_m, \mathfrak{u}_n) : n > m\} < \varepsilon\}.$$

In particular,

$$\forall\ \varepsilon > 0,\ \exists\ m_1 = m_1(\varepsilon) \in \mathbb{N},\ \forall\ m > m_1, \forall\ t \in \mathbb{N},\ \{J_\theta(\mathfrak{u}_m, \mathfrak{u}_{t+m}) < \varepsilon\}. \tag{12}$$

Let $i_0, j_0 \in \mathbb{N}, i_0 > j_0$, be fixed and arbitrary. Define

$$\mathfrak{e}_m = \mathfrak{u}_{i_0+m} \text{ and } \mathfrak{t}_m = \mathfrak{u}_{j_0+m} \text{ for } m \in \mathbb{N}. \tag{13}$$

Then, (12) gives

$$\lim_{m\to\infty} J_\theta(\mathfrak{u}_m, \mathfrak{e}_m) = \lim_{m\to\infty} J_\theta(\mathfrak{u}_m, \mathfrak{t}_m) = 0. \tag{14}$$

Therefore, by (11) and (13) and $(J_\theta 2)$, we obtain

$$\lim_{m\to\infty} \rho_\theta(\mathfrak{u}_m, \mathfrak{e}_m) = \lim_{m\to\infty} \rho_\theta(\mathfrak{u}_m, \mathfrak{t}_m) = 0. \tag{15}$$

Let $k = i_0 + m,\ l = j_0 + m$, then by using $(\rho_\theta 3)$, (13), and (15) we have

$$\begin{aligned}
\rho_\theta(\mathfrak{u}_k, \mathfrak{u}_l) = \rho_\theta(\mathfrak{u}_{i_0+m}, \mathfrak{u}_{j_0+m}) &\le \theta(\mathfrak{u}_{i_0+m}, \mathfrak{u}_{j_0+m})[\rho_\theta(\mathfrak{u}_{i_0+m}, \mathfrak{u}_m) + \rho_\theta(\mathfrak{u}_m, \mathfrak{u}_{j_0+m}]\\
&= \theta(\mathfrak{u}_{i_0+m}, \mathfrak{u}_{j_0+m})\rho_\theta(\mathfrak{u}_{i_0+m}, \mathfrak{u}_m) + \theta(\mathfrak{u}_{i_0+m}, \mathfrak{u}_{j_0+m})\rho_\theta(\mathfrak{u}_m, \mathfrak{u}_{j_0+m})\\
&= \theta(\mathfrak{u}_{i_0+m}, \mathfrak{u}_{j_0+m})\rho_\theta(\mathfrak{e}_m, \mathfrak{u}_m) + \theta(\mathfrak{u}_{i_0+m}, \mathfrak{u}_{j_0+m})\rho_\theta(\mathfrak{u}_m, \mathfrak{t}_m)\\
&\to 0 \text{ as } m \to \infty.
\end{aligned}$$

Thus, $\{\mathfrak{u}_n : n \in \{0\} \in \mathbb{N}\}$ is Cauchy. □

The following example validates Lemma 1.

Example 4. *Let $\mathfrak{U}^\star = [0,1]$ with the extended b-metric ρ_θ defined in Example 1. Let $\mathfrak{B}^\star = [0.6, 0.8]$ and $J_\theta : \mathfrak{U}^\star \times \mathfrak{U}^\star \to [0,\infty)$ be defined by*

$$J_\theta(\mathfrak{u}, \mathfrak{e}) = \begin{cases} \rho_\theta(\mathfrak{u}, \mathfrak{e}) & \text{if } \ \{\mathfrak{u}, \mathfrak{e}\} \subseteq \mathfrak{B}^\star \\ 4 & \text{if } \ \{\mathfrak{u}, \mathfrak{e}\} \nsubseteq \mathfrak{B}^\star. \end{cases}$$

Then, by Example 2, the mapping J_θ is an E.b-G pseudo-distance on $\mathfrak{U}^\star$. Define $\{\mathfrak{u}_n = 0.5 : n \in \mathbb{N}\}$, then $J_\theta(\mathfrak{u}_m, \mathfrak{u}_n) = \rho_\theta(\mathfrak{u}_m, \mathfrak{u}_n) = 0,\ \forall\ m, n \in \mathbb{N}$ and $\lim_{m\to\infty} \sup_{n>m} J_b(u_m, u_n) = 0$. Also $\lim_{n,m\to\infty} \theta(\mathfrak{u}_n, \mathfrak{u}_m) = 1.25$, which is a finite number. All the conditions of Lemma 1 hold, and $\lim_{n,m\to\infty} \rho_\theta(\mathfrak{u}_m, \mathfrak{u}_n) = \lim_{n,m\to\infty} \rho_\theta(0.5, 0.5) = 0$. Hence, $\{\mathfrak{u}_n = 0.5 : n \in \mathbb{N}\}$, is a Cauchy sequence.

Note. Throughout, let $\mathcal{H}^\star$ and $\mathcal{K}^\star$ denote the nonempty closed subsets of $\mathfrak{U}^\star$.

Lemma 2. *Let the space $(\mathfrak{U}^\star, \rho_\theta, J_\theta)$ be complete and $\Gamma : \mathcal{H}^\star \to 2^{\mathcal{K}^\star}$ be a multi-valued mapping. Then,*

$$\forall\ \mathfrak{u}, \mathfrak{e} \in \mathcal{H}^\star, \beta > 0\ \forall\ \mathfrak{t} \in \Gamma\mathfrak{u},\ \exists\ \mathfrak{x} \in \Gamma\mathfrak{e}\{J_\theta(\mathfrak{t}, \mathfrak{x}) \le H^{J_\theta}(\Gamma\mathfrak{u}, \Gamma\mathfrak{e}) + \beta\}. \tag{16}$$

Proof. Let $\mathfrak{u}, \mathfrak{e} \in \mathcal{H}^\star,\ \beta > 0$ and $\mathfrak{t} \in \Gamma\mathfrak{u}$ be fixed and arbitrary. Then by infimum definition,

$$\exists\ \mathfrak{x} \in \Gamma\mathfrak{e}, \{J_\theta(\mathfrak{t}, \mathfrak{x}) < \inf\{J_\theta(\mathfrak{t}, \mathfrak{y}) : \mathfrak{y} \in \Gamma\mathfrak{e}\} + \beta\}. \tag{17}$$

Next,

$$\begin{aligned}
\inf\{J_\theta(\mathfrak{t}, \mathfrak{y}) : \mathfrak{y} \in \Gamma\mathfrak{e}\} + \beta &\le \sup\{\inf\{J_\theta(\mathfrak{z}, \mathfrak{y}) : \mathfrak{y} \in \Gamma\mathfrak{e}\} : \mathfrak{z} \in \Gamma\mathfrak{u}\} + \beta\\
&\le \max\Big\{ \sup\{\inf\{J_\theta(\mathfrak{z}, \mathfrak{y}) : \mathfrak{y} \in \Gamma\mathfrak{e}\} : \mathfrak{z} \in \Gamma\mathfrak{u}\},\\
&\qquad \sup\{\inf\{J_\theta(\mathfrak{y}, \mathfrak{z}) : \mathfrak{z} \in \Gamma\mathfrak{u}\} : \mathfrak{y} \in \Gamma\mathfrak{e}\Big\} + \beta\\
&= H^{J_\theta}(\Gamma\mathfrak{u}, \Gamma\mathfrak{e}) + \beta.
\end{aligned}$$

Hence, by (17) we obtain

$$J_\theta(\mathfrak{t}, \mathfrak{x}) \le H^{J_\theta}(\Gamma\mathfrak{u}, \Gamma\mathfrak{e}) + \beta\}.$$

Thus, (16) holds. □

The following example validates Lemma 2.

Example 5. *Let* $(\mathfrak{U}^\star, \rho_\theta)$ *be the E.b-m space defined in Example 1. Let* $\mathcal{H}^\star = \{0.1, 0.2, 0.5, 0.6\}$, $\mathcal{K}^\star = [0.7, 0.9]$, *and* $\mathfrak{B}^\star = \{0.1, 0.2\} \cup [0.5, 0.9]$ *and* $J_\theta : \mathfrak{U}^\star \times \mathfrak{U}^\star \to [0, \infty)$ *be defined by*

$$J_\theta(\mathfrak{u}, \mathfrak{e}) = \begin{cases} d_\theta(\mathfrak{u}, \mathfrak{e}) & \text{if } \{\mathfrak{u}, \mathfrak{e}\} \subseteq \mathfrak{B}^\star \\ 51 & \text{if } \{\mathfrak{u}, \mathfrak{e}\} \nsubseteq \mathfrak{B}^\star \end{cases}$$

for all $\mathfrak{u}, \mathfrak{e} \in \mathfrak{U}^\star$. *Then, by Example 2, the mapping* J_θ *is an E.b-G pseudo-distance on* $\mathfrak{U}^\star$. *Assume that* $\Gamma : \mathcal{H}^\star \to 2^{\mathcal{K}^\star}$ *is of the form*

$$\Gamma\mathfrak{u} = \begin{cases} \{0.7, 0.75, 0.8\} & \text{if } \mathfrak{u} = 0.1 \\ \{0.8, 0.85, 0.9\} & \text{if } \mathfrak{u} = 0.2 \\ \{0.9\} & \text{if } \mathfrak{u} \in \{0.5, 0.6\}. \end{cases}$$

We consider the following cases.

Case (i) *If* $\Gamma\mathfrak{u} = \{0.7, 0.75, 0.8\}$, $\Gamma\mathfrak{e} = \{0.8, 0.85, 0.9\}$, *then* $\mathfrak{u} = 0.1$, $\mathfrak{e} = 0.2$ *and* $H^{J_\theta}(\Gamma\mathfrak{u}, \Gamma\mathfrak{e}) = 1.58$. *For* $\mathfrak{t} = 0.7 \in \Gamma\mathfrak{u}$, $\exists$, $\mathfrak{x} = 0.9 \in \Gamma\mathfrak{e}$, *such that for all* $\beta > 0$, *we have* $J_\theta(\mathfrak{t}, \mathfrak{x}) = J_\theta(0.7, 0.9) = 1.58 \leq H^{J_\theta}(\Gamma\mathfrak{u}, \Gamma\mathfrak{e}) = 1.58 + \beta$. *For* $\mathfrak{t} = 0.75 \in \Gamma\mathfrak{u}$, $\exists$, $\mathfrak{x} = 0.9 \in \Gamma\mathfrak{e}$, *such that for all* $\beta > 0$, *we have* $J_\theta(\mathfrak{t}, \mathfrak{x}) = J_\theta(0.75, 0.9) = 1.48 \leq H^{J_\theta}(\Gamma\mathfrak{u}, \Gamma\mathfrak{e}) = 1.58 + \beta$. *For* $\mathfrak{t} = 0.8 \in \Gamma\mathfrak{u}$, $\exists$, $\mathfrak{x} = 0.8 \in \Gamma\mathfrak{e}$, *such that for all* $\beta > 0$, *we have* $J_\theta(\mathfrak{t}, \mathfrak{x}) = J_\theta(0.8, 0.8) = 0 \leq H^{J_\theta}(\Gamma\mathfrak{u}, \Gamma\mathfrak{e}) = 1.58 + \beta$. *So in this case, Equation (16) holds.*

Case (ii) *If* $\Gamma\mathfrak{u} = \{0.7, 0.75, 0.8\}$, $\Gamma\mathfrak{e} = \{0.9\}$, *then* $\mathfrak{u} = 0.1$, $\mathfrak{e} \in \{0.5, 0.6\}$, $H^{J_\theta}(\Gamma\mathfrak{u}, \Gamma\mathfrak{e}) = 1.58$. *For* $\mathfrak{t} = 0.7 \in \Gamma\mathfrak{u}$, $\exists$, $\mathfrak{x} = 0.9 \in \Gamma\mathfrak{e}$, *such that for all* $\beta > 0$, *we have* $J_\theta(\mathfrak{t}, \mathfrak{x}) = J_\theta(0.7, 0.9) = 1.58 \leq H^{J_\theta}(\Gamma\mathfrak{u}, \Gamma\mathfrak{e}) = 1.58 + \beta$. *For* $\mathfrak{t} = 0.75 \in \Gamma\mathfrak{u}$, $\exists$, $\mathfrak{x} = 0.9 \in \Gamma\mathfrak{e}$, *such that for all* $\beta > 0$, *we have* $J_\theta(\mathfrak{t}, \mathfrak{x}) = J_\theta(0.75, 0.9) = 1.48 \leq H^{J_\theta}(\Gamma\mathfrak{u}, \Gamma\mathfrak{e}) = 1.58 + \beta$. *For* $\mathfrak{t} = 0.8 \in \Gamma\mathfrak{u}$, $\exists$, $\mathfrak{x} = 0.9 \in \Gamma\mathfrak{e}$, *such that for all* $\beta > 0$, *we have* $J_\theta(\mathfrak{t}, \mathfrak{x}) = J_\theta(0.8, 0.9) = 1.38 \leq H^{J_\theta}(\Gamma\mathfrak{u}, \Gamma\mathfrak{e}) = 1.58 + \beta$. *So in this case, Equation (16) holds.*

Case (iii) *If* $\Gamma\mathfrak{u} = \{0.8, 0.85, 0.9\}$, $\Gamma\mathfrak{e} = \{0.9\}$, *then* $\mathfrak{u} = 0.2$, $\mathfrak{e} \in \{0.5, 0.6\}$, *and* $H^{J_\theta}(\Gamma\mathfrak{u}, \Gamma\mathfrak{e}) = 1.38$. *For* $\mathfrak{t} = 0.8 \in \Gamma\mathfrak{u}$, $\exists$, $\mathfrak{x} = 0.9 \in \Gamma\mathfrak{e}$, *such that for all* $\beta > 0$, *we have* $J_\theta(\mathfrak{t}, \mathfrak{x}) = J_\theta(0.8, 0.9) = 1.38 \leq H^{J_\theta}(\Gamma\mathfrak{u}, \Gamma\mathfrak{e}) = 1.38 + \beta$. *For* $\mathfrak{t} = 0.85 \in \Gamma\mathfrak{u}$, $\exists$, $\mathfrak{x} = 0.9 \in \Gamma\mathfrak{e}$ *such that for all* $\beta > 0$, *we have* $J_\theta(\mathfrak{t}, \mathfrak{x}) = J_\theta(0.85, 0.9) = 1.3 \leq H^{J_\theta}(\Gamma\mathfrak{u}, \Gamma\mathfrak{e}) = 1.38 + \beta$. *For* $\mathfrak{t} = 0.9 \in \Gamma\mathfrak{u}$, $\exists$, $\mathfrak{x} = 0.9 \in \Gamma\mathfrak{e}$ *such that for all* $\beta > 0$, *we have* $J_\theta(\mathfrak{t}, \mathfrak{x}) = J_\theta(0.9, 0.9) = 0 \leq H^{J_\theta}(\Gamma\mathfrak{u}, \Gamma\mathfrak{e}) = 1.38 + \beta$. *So in this case, Equation (16) holds.*

In the following, we include our first main result.

Theorem 3. *Let the space* $(\mathfrak{U}^\star, \rho_\theta, J_\theta)$ *be complete, such that* J_θ *is associated with* $(\mathcal{H}^\star, \mathcal{K}^\star)$ *and* $(\mathcal{H}^\star, \mathcal{K}^\star)$ *has the* P^{J_θ}*-property. Let* $\Gamma : \mathcal{H}^\star \to 2^{\mathcal{K}^\star}$ *be a closed, multi-valued mapping, such that*

$$s_{\mathfrak{u}} H^{J_\theta}(\Gamma\mathfrak{u}, \Gamma\mathfrak{e}) \leq k J_\theta(\mathfrak{u}, \mathfrak{e}), \tag{18}$$

for all $\mathfrak{u}, \mathfrak{e} \in \mathfrak{U}^\star$, *and for some* $0 \leq k < 1$ *with*

$$\lim_{n,m\to\infty} \theta(\mathfrak{u}_n, \mathfrak{u}_m) < \frac{1}{k}, \quad \lim_{n,m\to\infty} \theta(\mathfrak{e}_n, \mathfrak{e}_m) < \frac{1}{k}.$$

Here, $s_{\mathfrak{u}} = \inf_{\mathfrak{e}\in\Gamma\mathfrak{u}} \theta(\mathfrak{u}, \mathfrak{e})$, $\mathfrak{u}_n \in \mathcal{H}_0$ *and* $\mathfrak{e}_n \in \Gamma\mathfrak{u}_n, n = 0, 1, 2, \cdots$. *Let* $\Gamma\mathfrak{u} \in CB(\mathfrak{U}^\star)$, $\Gamma\mathfrak{t} \subset \mathcal{K}_0$ *for each* $\mathfrak{u} \in \mathcal{H}^\star$, $\mathfrak{t} \in \mathcal{H}_0$. *Then* $D(\mathfrak{p}, \Gamma\mathfrak{p}) = dst(\mathcal{H}^\star, \mathcal{K}^\star)$ *for some* $\mathfrak{p} \in \mathcal{H}^\star$.

Proof. Let $\mathfrak{u}_0 \in \mathcal{H}_0, \mathfrak{e}_0 \in \Gamma\mathfrak{u}_0 \subseteq \mathcal{K}_0$. Then there exists $\mathfrak{u}_1 \in \mathcal{H}_0$, such that

$$J_\theta(\mathfrak{u}_1, \mathfrak{e}_0) = dst(\mathcal{H}^\star, \mathcal{K}^\star). \tag{19}$$

Since $\mathfrak{u}_0, \mathfrak{u}_1 \in \mathcal{H}_0, \mathfrak{e}_0 \in \Gamma\mathfrak{u}_0$, so that by Lemma 2, there exists $\mathfrak{e}_1 \in \Gamma\mathfrak{u}_1 \subseteq \mathcal{K}_0$, such that

$$J_\theta(\mathfrak{e}_0, \mathfrak{e}_1) \leq H^{J_\theta}(\Gamma\mathfrak{u}_0, \Gamma\mathfrak{u}_1) + k. \tag{20}$$

Again, as $\mathfrak{e}_1 \in \mathcal{K}_0$, there exists $\mathfrak{u}_2 \in \mathcal{H}_0$, such that

$$J_\theta(\mathfrak{u}_2, \mathfrak{e}_1) = dst(\mathcal{H}^\star, \mathcal{K}^\star). \tag{21}$$

Now $\mathfrak{u}_1, \mathfrak{u}_2 \in \mathcal{H}_0, \mathfrak{e}_1 \in \Gamma\mathfrak{u}_1$, so according to Lemma 2, there exists $\mathfrak{e}_2 \in \Gamma\mathfrak{u}_2$, such that

$$J_\theta(\mathfrak{e}_1, \mathfrak{e}_2) \leq H^{J_\theta}(\Gamma\mathfrak{u}_1, \Gamma\mathfrak{u}_2) + k^2. \tag{22}$$

We continue the above process, then by induction, we find $\{\mathfrak{u}_n : n \in \{0\} \cup \mathbb{N}\}$ and $\{\mathfrak{e}_n : n \in \{0\} \cup \mathbb{N}\}$, such that

(i) $\mathfrak{u}_n \in \mathcal{H}_0, \mathfrak{e}_n \in \mathcal{K}_0 \ \forall \ n \in \{0\} \cup \mathbb{N}$;
(ii) $\mathfrak{e}_n \in \Gamma\mathfrak{u}_n \ \forall \ n \in \{0\} \cup \mathbb{N}$;
(iii) $J_\theta(\mathfrak{u}_n, \mathfrak{e}_{n-1}) = dst(\mathcal{H}^\star, \mathcal{K}^\star) \ \forall \ n \in \mathbb{N}$;
(iv) $J_\theta(\mathfrak{e}_{n-1}, \mathfrak{e}_n) \leq H^{J_\theta}(\Gamma\mathfrak{u}_{n-1}, \Gamma\mathfrak{u}_n) + k^n \ \forall \ n \in \mathbb{N}$.

Now, for any $n \in \mathbb{N}$, we have $J_\theta(\mathfrak{u}_n, \mathfrak{e}_{n-1}) = dst(\mathcal{H}^\star, \mathcal{K}^\star)$ and $J_\theta(\mathfrak{u}_{n+1}, \mathfrak{e}_n) = dst(\mathcal{H}^\star, \mathcal{K}^\star)$. Since the pair $(\mathcal{H}^\star, \mathcal{K}^\star)$ has the P^{J_θ}-property, we deduce

$$J_\theta(\mathfrak{u}_n, \mathfrak{u}_{n+1}) = J_\theta(\mathfrak{e}_{n-1}, \mathfrak{e}_n), \ \forall \ n \in \mathbb{N}. \tag{23}$$

Now for $\mathfrak{u} = \mathfrak{u}_n, \mathfrak{e} = \mathfrak{u}_{n+1},$ and $n \in \mathbb{N}$ by (18), we obtain

$$H^{J_\theta}(\Gamma\mathfrak{u}_n, \Gamma\mathfrak{u}_{n+1}) \leq \frac{k}{s_{\mathfrak{u}_n}} J_\theta(\mathfrak{u}_n, \mathfrak{u}_{n+1}) \ \forall \ n \in \{0\} \cup \mathbb{N}. \tag{24}$$

Next, for all $n \in \mathbb{N}$, we have

$$\begin{aligned}
J_\theta(\mathfrak{u}_n, \mathfrak{u}_{n+1}) &= J_\theta(\mathfrak{e}_{n-1}, \mathfrak{e}_n) \\
&\leq H^{J_\theta}(\Gamma\mathfrak{u}_{n-1}, \Gamma\mathfrak{u}_n) + k^n \\
&\leq \frac{k}{s_{\mathfrak{u}_{n-1}}} J_\theta(\mathfrak{u}_{n-1}, \mathfrak{u}_n) + k^n \\
&\leq k J_\theta(\mathfrak{u}_{n-1}, \mathfrak{u}_n) + k^n \\
&= k J_\theta(\mathfrak{e}_{n-2}, \mathfrak{e}_{n-1}) + k^n \\
&\leq k\left[H^{J_\theta}(\Gamma\mathfrak{u}_{n-2}, \Gamma\mathfrak{u}_{n-1}) + k^{n-1}\right] + k^n \\
&= k H^{J_\theta}(\Gamma\mathfrak{u}_{n-2}, \Gamma\mathfrak{u}_{n-1}) + 2k^n \\
&\leq \frac{k^2}{s_{\mathfrak{u}_{n-2}}} J_\theta(\mathfrak{u}_{n-2}, \mathfrak{u}_{n-1}) + 2k^n \\
&\leq k^2 J_\theta(\mathfrak{u}_{n-2}, \mathfrak{u}_{n-1}) + 2k^n \\
&= k^2 J_\theta(\mathfrak{e}_{n-3}, \mathfrak{e}_{n-2}) + 2k^n \\
&\leq k^2\left[H^{J_\theta}(\Gamma\mathfrak{u}_{n-3}, \Gamma\mathfrak{u}_{n-2}) + k^{n-2}\right] + 2k^n \\
&= k^2 H^{J_\theta}(\Gamma\mathfrak{u}_{n-3}, \Gamma\mathfrak{u}_{n-2}) + 3k^n \\
&\leq \frac{k^3}{s_{\mathfrak{u}_{n-3}}} J_\theta(\mathfrak{u}_{n-3}, \mathfrak{u}_{n-2}) + 3k^n \\
&\leq k^3 J_\theta(\mathfrak{u}_{n-3}, \mathfrak{u}_{n-2}) + 3k^n \\
&\leq \cdots \leq k^n J_\theta(\mathfrak{u}_0, \mathfrak{u}_1) + nk^n.
\end{aligned}$$

We obtain

$$J_\theta(\mathfrak{u}_n, \mathfrak{u}_{n+1}) \leq k^n J_\theta(\mathfrak{u}_0, \mathfrak{u}_1) + nk^n \ \forall\ n \in \mathbb{N}.$$

Now, for each $n > m$, we obtain

$$\begin{aligned}
J_\theta(\mathfrak{u}_m, \mathfrak{u}_n) &\leq \theta(\mathfrak{u}_m, \mathfrak{u}_n)[J_\theta(\mathfrak{u}_m, \mathfrak{u}_{m+1}) + J_\theta(\mathfrak{u}_{m+1}, \mathfrak{u}_n)] \\
&\leq \theta(\mathfrak{u}_m, \mathfrak{u}_n)J_\theta(\mathfrak{u}_m, \mathfrak{u}_{m+1}) + \theta(\mathfrak{u}_m, \mathfrak{u}_n)\theta(\mathfrak{u}_{m+1}, \mathfrak{u}_n)J_\theta(\mathfrak{u}_{m+1}, \mathfrak{u}_{m+2}) \\
&\quad + \theta(\mathfrak{u}_m, \mathfrak{u}_n)\theta(\mathfrak{u}_{m+1}, \mathfrak{u}_n)J_\theta(\mathfrak{u}_{m+2}, \mathfrak{u}_n) \\
&\leq \cdots \leq \sum_{i=m}^{n-1}\left(\prod_{j=m}^{i}\theta(\mathfrak{u}_j, \mathfrak{u}_n)J_\theta(\mathfrak{u}_i, \mathfrak{u}_{i+1})\right) \\
&\leq \sum_{i=m}^{n-1}\left(\prod_{j=m}^{i}\theta(\mathfrak{u}_j, \mathfrak{u}_n)(k^i J_\theta(\mathfrak{u}_0, \mathfrak{u}_1) + ik^i)\right) \\
&= \sum_{i=m}^{n-1}\left(\prod_{j=m}^{i}\theta(\mathfrak{u}_j, \mathfrak{u}_n)k^i J_\theta(\mathfrak{u}_0, \mathfrak{u}_1) + \prod_{j=m}^{i}\theta(\mathfrak{u}_j, \mathfrak{u}_n)ik^i\right) \\
&= \sum_{i=m}^{n-1}\left(\prod_{j=m}^{i}\theta(\mathfrak{u}_j, \mathfrak{u}_n)k^i J_\theta(\mathfrak{u}_0, \mathfrak{u}_1)\right) + \sum_{i=m}^{n-1}\left(\prod_{j=m}^{i}\theta(\mathfrak{u}_j, \mathfrak{u}_n)ik^i\right) \\
&\leq \sum_{i=m}^{n-1}\left(\prod_{j=1}^{i}\theta(\mathfrak{u}_j, \mathfrak{u}_n)k^i J_\theta(\mathfrak{u}_0, \mathfrak{u}_1)\right) + \sum_{i=m}^{n-1}\left(\prod_{j=1}^{i}\theta(\mathfrak{u}_j, \mathfrak{u}_n)ik^i\right) \\
&= J_\theta(\mathfrak{u}_0, \mathfrak{u}_1)\sum_{i=m}^{n-1}\left(\prod_{j=1}^{i}\theta(\mathfrak{u}_j, \mathfrak{u}_n)k^i\right) + \sum_{i=m}^{n-1}\left(\prod_{j=1}^{i}\theta(\mathfrak{u}_j, \mathfrak{u}_n)ik^i\right). \qquad (25)
\end{aligned}$$

Let, $a_m = \prod_{j=1}^{m}\theta(\mathfrak{u}_j, \mathfrak{u}_n)k^m,\ S_m = \sum_{i=1}^{m}\left(\prod_{j=1}^{i}\theta(\mathfrak{u}_j, \mathfrak{u}_n)k^i\right)$ and $a'_m = \prod_{j=1}^{m}\theta(\mathfrak{u}_m, \mathfrak{u}_n)mk^m$, $S'_m = \sum_{i=1}^{m}\left(\prod_{j=1}^{i}\theta(\mathfrak{u}_j, \mathfrak{u}_n)ik^i\right)$. Then $\lim_{m\to\infty}\frac{a_{m+1}}{a_m} = \lim_{m,n\to\infty}\theta(\mathfrak{u}_{m+1}, \mathfrak{u}_n)k < 1$, and

$$\begin{aligned}
\lim_{m\to\infty}\frac{a'_{m+1}}{a'_m} &= \lim_{m,n\to\infty}\theta(\mathfrak{u}_{m+1}, \mathfrak{u}_n)\left(\frac{m+1}{m}\right)k \\
&= \lim_{m,n\to\infty}\theta(\mathfrak{u}_{m+1}, \mathfrak{u}_n)\left(1 + \frac{1}{m}\right)k \\
&= \lim_{m,n\to\infty}\theta(\mathfrak{u}_{m+1}, \mathfrak{u}_n)k + \lim_{m,n\to\infty}\theta(\mathfrak{u}_{m+1}, \mathfrak{u}_n)\frac{k}{m} \\
&< 1 + 0.
\end{aligned}$$

Since $\lim_{m,n\to\infty}\theta(\mathfrak{u}_{m+1}, \mathfrak{u}_n)$ is finite and $\lim_{m\to\infty}\frac{k}{m} = 0$, the series $\sum_{i=1}^{\infty}\left(\prod_{j=1}^{i}\theta(\mathfrak{u}_j, \mathfrak{u}_n)k^i\right)$ and $\sum_{i=1}^{\infty}\left(\prod_{j=1}^{i}\theta(\mathfrak{u}_j, \mathfrak{u}_n)ik^i\right)$ converge by ratio test for each $n \in \mathbb{N}$. For $n > m$, inequality (25) implies

$$J_\theta(\mathfrak{u}_m, \mathfrak{u}_n) \leq J_\theta(\mathfrak{u}_0, \mathfrak{u}_1)[S_{n-1} - S_m] + S'_{n-1} - S'_m.$$

Letting $m \to \infty$, we obtain

$$\lim_{m\to\infty}\sup_{n>m} J_\theta(\mathfrak{u}_m, \mathfrak{u}_n) = 0. \qquad (26)$$

From inequalities (23) and (24), we have

$$\lim_{m\longrightarrow\infty}\sup_{n>m} J_\theta(\mathfrak{e}_m, \mathfrak{e}_n) = 0.$$

Also $\lim_{n,m\to\infty}\theta(\mathfrak{u}_n,\mathfrak{u}_m)<\frac{1}{k}$, and $\lim_{n,m\to\infty}\theta(\mathfrak{e}_n,\mathfrak{e}_m)<\frac{1}{k}$. By Lemma 1, the sequence $\{\mathfrak{u}_n : n\in\{0\}\cup\mathbb{N}\}$ is Cauchy in $\mathcal{H}^\star$ and $\{\mathfrak{e}_n : n\in\{0\}\cup\mathbb{N}\}$ is Cauchy in $\mathcal{K}^\star$. Since the subsets $\mathcal{H}^\star$ and $\mathcal{K}^\star$ are closed in the complete space $(\mathfrak{U}^\star,\rho_\theta)$, there is some $\mathfrak{p}$ in $\mathcal{H}^\star$ and $\mathfrak{q}$ in $\mathcal{K}^\star$ such that $\mathfrak{u}_n\to\mathfrak{p},\ \mathfrak{e}_n\to\mathfrak{q}$. Since $\mathfrak{e}_n\in\Gamma\mathfrak{u}_n$ for all $n\in\{0\}\cup\mathbb{N}$ and the multi-valued non-self mapping Γ is closed, $\mathfrak{q}\in\Gamma\mathfrak{p}$. Since $J_\theta(\mathfrak{u}_n,\mathfrak{e}_{n-1})=dst(\mathcal{H}^\star,\mathcal{K}^\star)$ and J_θ is associated with $(\mathcal{H}^\star,\mathcal{K}^\star)$, we deduce that $\rho_\theta(\mathfrak{p},\mathfrak{q})=dst(\mathcal{H}^\star,\mathcal{K}^\star)$. Now,

$$dst(\mathcal{H}^\star,\mathcal{K}^\star)=\rho_\theta(\mathfrak{p},\mathfrak{q})\geq D(\mathfrak{p},\Gamma\mathfrak{p})\geq D(\mathfrak{p},\mathcal{K}^\star)\geq dst(\mathcal{H}^\star,\mathcal{K}^\star),$$

Hence,

$$D(\mathfrak{p},\Gamma\mathfrak{p})=dst(\mathcal{H}^\star,\mathcal{K}^\star).$$

This completes the proof. □

In the following, we include our second result.

Theorem 4. *Let the space $(\mathfrak{U}^\star,\rho_\theta)$ be complete and the pair $(\mathcal{H}^\star,\mathcal{K}^\star)$ has the P^{ρ_θ}-property. Let $\Gamma:\mathcal{H}^\star\to 2^{\mathcal{K}^\star}$ be a closed multi-valued mapping that satisfies*

$$s_{\mathfrak{u}}H^{\rho_\theta}(\Gamma\mathfrak{u},\Gamma\mathfrak{e})\leq k\rho_\theta(\mathfrak{u},\mathfrak{e}),\tag{27}$$

for all $\mathfrak{u},\mathfrak{e}\in\mathfrak{U}^\star$ and for some $0\leq k<1$ with

$$\lim_{n,m\to\infty}\theta(\mathfrak{u}_n,\mathfrak{u}_m)<\frac{1}{k},\quad \lim_{n,m\to\infty}\theta(\mathfrak{e}_n,\mathfrak{e}_m)<\frac{1}{k}.$$

Here, $s_{\mathfrak{u}}=\inf_{\mathfrak{e}\in\Gamma\mathfrak{u}}\theta(\mathfrak{u},\mathfrak{e})$, $\mathfrak{u}_n\in\mathcal{H}_0$ and $\mathfrak{e}_n\in\Gamma\mathfrak{u}_n, n=0,1,2\cdots$. Let $\Gamma\mathfrak{u}\in CB(\mathfrak{U}^\star)$, $\Gamma\mathfrak{t}\subset\mathcal{K}_0$ for each $\mathfrak{u}\in\mathcal{H}^\star,\ \mathfrak{t}\in\mathcal{H}_0$. Then $D(\mathfrak{p},\Gamma\mathfrak{p})=dst(\mathcal{H}^\star,\mathcal{K}^\star)$ for some $\mathfrak{p}\in\mathcal{H}^\star$.

Proof. Since every extended b-metric ρ_θ is an E.b-G pseudo-distance on $\mathfrak{U}^\star$, if $J_\theta=\rho_\theta$, then ρ_θ is associated with each pair $(\mathcal{H}^\star,\mathcal{K}^\star)$, because of the continuity of ρ_θ and P^{J_θ}-property becomes P^{ρ_θ}-property. Thus, all the axioms of Theorem 3 are fulfilled. Hence, there exists some $\mathfrak{p}\in\mathcal{H}^\star$ such that $D(\mathfrak{p},\Gamma\mathfrak{p})=dst(\mathcal{H}^\star,\mathcal{K}^\star)$. (The detailed proof of Theorem 4 is given in Appendix A). □

The following is another result for complete space $(\mathfrak{U}^\star,\rho_\theta,J_\theta)$.

Theorem 5. *Let the space $(\mathfrak{U}^\star,\rho_\theta,J_\theta)$ be complete space such that J_θ is associated with the pair $(\mathcal{H}^\star,\mathcal{K}^\star)$ and $(\mathcal{H}^\star,\mathcal{K}^\star)$ has the P^{J_θ}-property. Let $\Gamma:\mathcal{H}^\star\to\mathcal{K}^\star$ be a non-self continuous mapping that satisfies*

$$s_{\mathfrak{u}}J_\theta(\Gamma\mathfrak{u},\Gamma\mathfrak{e})\leq\ kJ_\theta(\mathfrak{u},\mathfrak{e}),\tag{28}$$

for all $\mathfrak{u},\mathfrak{e}\in\mathcal{H}^\star$ and for some $0\leq k<1$ with

$$\lim_{n,m\to\infty}\theta(\mathfrak{u}_n,\mathfrak{u}_m)<\frac{1}{k},\quad \lim_{n,m\to\infty}\theta(\mathfrak{e}_n,\mathfrak{e}_m)<\frac{1}{k}.$$

Here, $s_{\mathfrak{u}}=\theta(\mathfrak{u},\Gamma\mathfrak{u})$, $\mathfrak{u}_n\in\mathcal{H}_0$, $\mathfrak{e}_n=\Gamma\mathfrak{u}_n, n=0,1,2\cdots$. Let $\Gamma\mathfrak{t}\in\mathcal{K}_0$ for each $\mathfrak{t}\in\mathcal{H}_0$. Then $\rho_\theta(\mathfrak{p},\Gamma\mathfrak{p})=dst(\mathcal{H}^\star,\mathcal{K}^\star)$ for some $\mathfrak{p}\in\mathcal{H}^\star$.

Proof. Since the contraction mapping $\Gamma:\mathcal{H}^\star\to\mathcal{K}^\star$ is continuous, if $\{\mathfrak{u}_n : n\in\mathbb{N}\}$ and $\{\mathfrak{e}_n : n\in\mathbb{N}\}$ are any two sequences in $\mathcal{H}^\star$ and $\mathcal{K}^\star$, respectively, such that $\mathfrak{u}_n\to\ \mathfrak{p}\in\mathcal{H}^\star$, and $\mathfrak{e}_n\to\mathfrak{q}\in\mathcal{K}^\star$, and $\mathfrak{e}_n=\Gamma\mathfrak{u}_n\ \forall\ n\in\mathbb{N}$, then $\Gamma\mathfrak{u}_n\to\Gamma\mathfrak{p}$ which implies $\mathfrak{e}_n\to\Gamma\mathfrak{u}$. Since the limit of sequence in $(\mathfrak{U}^\star,\rho_\theta)$ is unique, $\mathfrak{p}=\Gamma\mathfrak{q}$. Thus, the mapping Γ is closed. All the axioms of Theorem 3 are fulfilled. Hence, there exists some $\mathfrak{p}\in\mathcal{H}^\star$ such that $D(\mathfrak{p},\Gamma\mathfrak{p})=dst(\mathcal{H}^\star,\mathcal{K}^\star)$. (The detailed proof of Theorem 5 is given in Appendix B). □

Theorem 6. *Let the space* $(\mathfrak{U}^\star, \rho_\theta)$ *be complete and the pair* $(\mathcal{H}^\star, \mathcal{K}^\star)$ *has the* P^{ρ_θ}*-property. Let* $\Gamma : \mathcal{H}^\star \to \mathcal{K}^\star$ *be a non-self continuous mapping that satisfies*

$$s_{\mathfrak{u}}\rho_\theta(\Gamma\mathfrak{u}, \Gamma\mathfrak{e}) \leq k\rho_\theta(\mathfrak{u}, \mathfrak{e}), \tag{29}$$

for all $\mathfrak{u}, \mathfrak{e} \in \mathcal{H}^\star$*, and for some* $0 \leq k < 1$ *with*

$$\lim_{n,m\to\infty} \theta(\mathfrak{u}_n, \mathfrak{u}_m) < \frac{1}{k}, \quad \lim_{n,m\to\infty} \theta(\mathfrak{e}_n, \mathfrak{e}_m) < \frac{1}{k}.$$

Here, $s_{\mathfrak{u}} = \theta(\mathfrak{u}, \Gamma\mathfrak{u})$*,* $\mathfrak{u}_n \in \mathcal{H}_0$ *and* $\mathfrak{e}_n = \Gamma\mathfrak{u}_n, n = 0, 1, 2\cdots$*. Let* $\Gamma\mathfrak{t} \in \mathcal{K}_0$ *for each* $\mathfrak{t} \in \mathcal{H}_0$*. Then* $\rho_\theta(\mathfrak{p}, \Gamma\mathfrak{p}) = dst(\mathcal{H}^\star, \mathcal{K}^\star)$ *for some* $\mathfrak{p} \in \mathcal{H}^\star$*.*

Proof. By setting $J_\theta = \rho_\theta$ in Theorem 5, we arrive at the desired result. (The detailed proof of Theorem 6 is given in Appendix C). □

4. Consequences and Examples

In this section, we include some important B.P. point theorems in the settings of b-m space $(\mathfrak{U}^\star, \rho_b)$ and b-G pseudo-distance space $(\mathfrak{U}^\star, \rho_b, J_b)$. We also furnish readers with concrete examples to validate our results.

Corollary 1. *Let the space* $(\mathfrak{U}^\star, \rho_b, J_b)$ *be complete such that* J_b *is associated with the pair* $(\mathcal{H}^\star, \mathcal{K}^\star)$ *and* $(\mathcal{H}^\star, \mathcal{K}^\star)$ *has the* P^{J_b}*-property. Let* $\Gamma : \mathcal{H}^\star \to 2^{\mathcal{K}^\star}$ *be a multi-valued closed mapping that satisfies*

$$sH^{J_b}(\Gamma\mathfrak{u}, \Gamma\mathfrak{e}) \leq kJ_b(\mathfrak{u}, \mathfrak{e}), \tag{30}$$

for all $\mathfrak{u}, \mathfrak{e} \in \mathcal{H}^\star$ *and for some* $0 \leq k < 1$ *with* $ks < 1$*. Let* $\Gamma\mathfrak{u} \in CB(\mathfrak{U}^\star)$*,* $\Gamma\mathfrak{t} \subset \mathcal{K}_0$ *for each* $\mathfrak{u} \in \mathcal{H}^\star$*,* $\mathfrak{t} \in \mathcal{H}_0$*. Then* $D(\mathfrak{p}, \Gamma\mathfrak{p}) = dst(\mathcal{H}^\star, \mathcal{K}^\star)$ *for some* $\mathfrak{p} \in \mathcal{H}^\star$*.*

Proof. By setting $J_\theta = J_b$ in Theorem 3, we arrive at the desired result. □

Corollary 2. *Let the space* $(\mathfrak{U}^\star, \rho_b)$ *be complete and the pair* $(\mathcal{H}^\star, \mathcal{K}^\star)$ *has the* P^{ρ_b}*-property. Let* $\Gamma : \mathcal{H}^\star \to 2^{\mathcal{K}^\star}$ *be a closed multi-valued mapping that satisfies*

$$sH^{\rho_b}(\Gamma\mathfrak{u}, \Gamma\mathfrak{e}) \leq k\rho_b(\mathfrak{u}, \mathfrak{e}), \tag{31}$$

for all $\mathfrak{u}, \mathfrak{e} \in \mathfrak{U}^\star$ *and for some* $0 \leq k < 1$ *with* $ks < 1$*. Let* $\Gamma\mathfrak{u} \in CB(\mathfrak{U}^\star)$*,* $\Gamma\mathfrak{t} \subset \mathcal{K}_0$ *for each* $\mathfrak{u} \in \mathcal{H}^\star$*,* $\mathfrak{t} \in \mathcal{H}_0$*. Then* $D(\mathfrak{p}, \Gamma\mathfrak{p}) = dst(\mathcal{H}^\star, \mathcal{K}^\star)$ *for some* $\mathfrak{p} \in \mathcal{H}^\star$*.*

Proof. By setting $\rho_\theta = d_b$ in Theorem 4, we arrive at the desired result. □

Corollary 3. *Let the space* $(\mathfrak{U}^\star, \rho)$ *be complete space and the pair* $(\mathcal{H}^\star, \mathcal{K}^\star)$ *has the* P^ρ*-property. Let* $\Gamma : \mathcal{H}^\star \to 2^{\mathcal{K}^\star}$ *be a closed, multi-valued mapping that satisfies*

$$H^\rho(\Gamma\mathfrak{u}, \Gamma\mathfrak{e}) \leq k\rho(\mathfrak{u}, \mathfrak{e}), \tag{32}$$

for all $\mathfrak{u}, \mathfrak{e} \in \mathcal{H}^\star$ *and for some* $0 \leq k < 1$*. Let* $\Gamma\mathfrak{u} \in CB(\mathfrak{U}^\star)$*,* $\Gamma\mathfrak{t} \subset \mathcal{K}_0$ *for each* $\mathfrak{u} \in \mathcal{H}^\star$*,* $\mathfrak{t} \in \mathcal{H}_0$*. Then* $D(\mathfrak{p}, \Gamma\mathfrak{p}) = dst(\mathcal{H}^\star, \mathcal{K}^\star)$ *for some* $\mathfrak{p} \in \mathcal{H}^\star$*.*

Proof. By setting $\rho_\theta = \rho$ in Theorem 4, we arrive at the desired result. □

Corollary 4. *Let the space* $(\mathfrak{U}^\star, \rho_b, J_b)$ *be complete such that* J_b *is associated with the pair* $(\mathcal{H}^\star, \mathcal{K}^\star)$ *and* $(\mathcal{H}^\star, \mathcal{K}^\star)$ *has the* P^{J_b}*-property. Let* $\Gamma : \mathcal{H}^\star \to \mathcal{K}^\star$ *be a non-self continuous mapping that satisfies*

$$sJ_b(\Gamma\mathfrak{u}, \Gamma\mathfrak{e}) \leq kJ_b(\mathfrak{u}, \mathfrak{e}), \tag{33}$$

for all $\mathfrak{u}, \mathfrak{e} \in \mathcal{H}^\star$ *and for some* $0 \leq k < 1$ *with* $ks < 1$. *Let* $\Gamma\mathfrak{t} \in \mathcal{K}_0$ *for each* $\mathfrak{t} \in \mathcal{H}_0$. *Then* $\rho_b(\mathfrak{p}, \Gamma\mathfrak{p}) = dst(\mathcal{H}^\star, \mathcal{K}^\star)$ *for some* $\mathfrak{p} \in \mathcal{H}^\star$.

Proof. By setting $J_\theta = J_b$ in Theorem 5, we arrive at the desired result. □

Corollary 5. *Let the space* $(\mathfrak{U}^\star, \rho_b)$ *be complete and the pair* $(\mathcal{H}^\star, \mathcal{K}^\star)$ *has the* P^{ρ_b}*-property. Let* $\Gamma : \mathcal{H}^\star \to \mathcal{K}^\star$ *be a non-self continuous mapping that satisfies*

$$s\rho_b(\Gamma\mathfrak{u}, \Gamma\mathfrak{e}) \leq k\rho_b(\mathfrak{u}, \mathfrak{e}), \tag{34}$$

for all $\mathfrak{u}, \mathfrak{e} \in \mathcal{H}^\star$ *and for some* $0 \leq k < 1$ *with* $ks < 1$. *Let* $\Gamma\mathfrak{t} \in \mathcal{K}_0$ *for each* $\mathfrak{t} \in \mathcal{H}_0$. *Then* $\rho_b(\mathfrak{p}, \Gamma\mathfrak{p}) = dst(\mathcal{H}^\star, \mathcal{K}^\star)$ *for some* $\mathfrak{p} \in \mathcal{H}^\star$.

Proof. By setting $\rho_\theta = \rho_b$ in Theorem 6, we arrive at the desired result. □

The following example validates Theorem 3.

Example 6. *Let* $\mathfrak{U}^\star = [0,1]$ *with the extended b-metric* ρ_θ *defined in Example 1. Let* $\mathcal{H}^\star = [0.1, 0.2]$, $\mathcal{K}^\star = [0.3, 0.4]$ *and* $\mathfrak{B}^\star = [0.1, 0.125] \cup [0.2, 0.4]$ *and* $J_\theta : \mathfrak{U}^\star \times \mathfrak{U}^\star \to [0, \infty)$ *be defined by*

$$J_\theta(\mathfrak{u}, \mathfrak{e}) = \begin{cases} \rho_\theta(\mathfrak{u}, \mathfrak{e}) & \text{if } \{\mathfrak{u}, \mathfrak{e}\} \subseteq \mathfrak{B}^\star \\ 110 & \text{if } \{\mathfrak{u}, \mathfrak{e}\} \nsubseteq \mathfrak{B}^\star. \end{cases}$$

Then, by Example 2, the mapping J_θ *is an E.b-G pseudo-distance on* $\mathfrak{U}^\star$. *Define* $\Gamma : \mathcal{H}^\star \to 2^{\mathcal{K}^\star}$ *as*

$$\Gamma\mathfrak{u} = \begin{cases} \{0.3\} \cup [0.375, 0.4] & \text{for } \mathfrak{u} \in [0.1, 0.125] \\ [0.375, 0.4] & \text{for } \mathfrak{u} \in (0.125, 0.15) \\ [0.35, 0.4] & \text{for } \mathfrak{u} \in [0.15, 0.175) \\ [0.325, 0.4] & \text{for } \mathfrak{u} \in [0.175, 0.1875) \\ \{0.3\} \cup [0.325, 0.4] & \text{for } \mathfrak{u} = 0.1875 \\ \{0.4\} & \text{for } \mathfrak{u} \in (0.1875, 0.2]. \end{cases}$$

(1) *We show that the pair* $(\mathcal{H}^\star, \mathcal{K}^\star)$ *has the* P^{J_θ}*-property.*
Observe that $dst(\mathcal{H}^\star, \mathcal{K}^\star) = 12.5$ *and* $\mathcal{H}_0 = \{0.2\}$, $= \{0.4\}$.
Hence, $(\mathcal{H}^\star, \mathcal{K}^\star)$ *has the* P^{J_θ}*-property.*
Also, $\Gamma(\mathcal{H}_0) = \Gamma(0.2) = 0.4 \in \mathcal{K}_0$.

(2) *We show that the mapping* J_θ *is associated with* $(\mathcal{H}^\star, \mathcal{K}^\star)$.
Let $\{\mathfrak{u}_n : n \in \mathbb{N}\}$ *and* $\{\mathfrak{e}_n : n \in \mathbb{N}\}$ *be any two sequences in* $\mathfrak{U}^\star$ *such that* $\lim_{n\to\infty} \mathfrak{u}_n = \mathfrak{u}$, $\lim_{n\to\infty} \mathfrak{e}_n = \mathfrak{e}$ *and*

$$J_\theta(\mathfrak{u}_n, \mathfrak{e}_{n-1}) = dst(\mathcal{H}^\star, \mathcal{K}^\star) \text{ for all } n \in \mathbb{N}. \tag{35}$$

Since $dst(\mathcal{H}^\star, \mathcal{K}^\star) = 12.5 < 110$. *By definition of* J_θ *we have*

$$J_\theta(\mathfrak{u}_n, \mathfrak{e}_{n-1}) = \rho_\theta(\mathfrak{u}_n, \mathfrak{e}_{n-1}) = dst(\mathcal{H}^\star, \mathcal{K}^\star). \tag{36}$$

By (36) and continuity of ρ_θ *we have* $\rho_\theta(\mathfrak{u}, \mathfrak{e}) = dst(\mathcal{H}^\star, \mathcal{K}^\star)$.

(3) *We show that (18) holds, i.e.,*

$$s_{\mathfrak{u}} H^{J_\theta}(\Gamma\mathfrak{u}, \Gamma\mathfrak{e}) \leq kJ_\theta(\mathfrak{u}, \mathfrak{e}) \text{ for all } \mathfrak{u}, \mathfrak{e} \in \mathcal{H}^\star \text{ and for some } k \in [0, 1). \tag{37}$$

Let $\mathfrak{u}, \mathfrak{e} \in \mathcal{H}^\star$ *be arbitrary and fixed, and* $k = \frac{1}{2} \in [0, 1)$. *By definition of* Γ, *we have* $\Gamma(\mathcal{H}^\star) \subset \mathcal{K}^\star = [0.3, 0.4] \subset \mathfrak{B}^\star$. *Moreover, by definition of* J_θ *we have* $H^{J_\theta}(\Gamma\mathfrak{u}, \Gamma\mathfrak{e}) \leq 12$, *for each* $\mathfrak{u}, \mathfrak{e} \in \mathcal{H}^\star$. *We discuss the following cases.*

Case(a) *If* $\{\mathfrak{u}, \mathfrak{e}\} \cap \mathfrak{B}^\star \neq \{\mathfrak{u}, \mathfrak{e}\}$, *then there are three possibilities.*

(i) $\mathfrak{u} \notin \mathfrak{B}^\star$ *and* $\mathfrak{e} \notin \mathfrak{B}^\star$.

(ii) $\mathfrak{u} \notin \mathfrak{B}^\star$ *and* $\mathfrak{e} \in \mathfrak{B}^\star$.

(iii) $\mathfrak{u} \in \mathfrak{B}^\star$ *and* $\mathfrak{e} \notin \mathfrak{B}^\star$.

If $\mathfrak{u} \notin \mathfrak{B}^\star$, *then* $\mathfrak{u} \in (0.125, 0.2)$ *and* $s_{\mathfrak{u}} \leq 3 \; \forall \; \mathfrak{u} \in (0.125, 0.2)$, $J_\theta(\mathfrak{u}, \mathfrak{e}) = 110$ *and* $s_{\mathfrak{u}} H^{J_\theta}(\Gamma\mathfrak{u}, \Gamma\mathfrak{e}) \leq 3(12) \leq \frac{110}{2} = kJ_\theta(\mathfrak{u}, \mathfrak{e})$.
If $\mathfrak{u} \in \mathfrak{B}^\star$, *then* $\mathfrak{u} \in \mathcal{H}^\star \cap \mathfrak{B}^\star = [0.1, 0.125] \cup \{0.2\}$ *and* $s_{\mathfrak{u}} \leq 3 \; \forall \; \mathfrak{u} \in [0.1, 0.125] \cup \{0.2\}$, $J_\theta(\mathfrak{u}, \mathfrak{e}) = 110$ *and* $s_{\mathfrak{u}} H^{J_\theta}(\Gamma\mathfrak{u}, \Gamma\mathfrak{e}) \leq 3(12) \leq \frac{110}{2} = kJ_\theta(\mathfrak{u}, \mathfrak{e})$. *So (37) holds.*

Case(b) *If* $\{\mathfrak{u}, \mathfrak{e}\} \cap \mathfrak{B}^\star = \{\mathfrak{u}, \mathfrak{e}\}$, *then* $\mathfrak{u}, \mathfrak{e} \in \mathcal{H}^\star \cap \mathfrak{B}^\star = [0.1, 0.125] \cup \{0.2\}$. *Now, from the definition of* Γ, *we have:* $\Gamma\mathfrak{u} = \Gamma\mathfrak{e} \; \forall \; \mathfrak{u}, \mathfrak{e} \in [0.1, 0.125]$, $\Gamma 0.2 \subset \Gamma\mathfrak{u} \; \forall \; \mathfrak{u} \in [0.1, 0.125]$. *Thus* $H^{J_\theta}(\Gamma\mathfrak{u}, \Gamma\mathfrak{e}) = 0$ *for all* $\mathfrak{u}, \mathfrak{e} \in [0.1, 0.125] \cup \{0.2\}$.
Hence, $s_{\mathfrak{u}} H^{J_\theta}(\Gamma\mathfrak{u}, \Gamma\mathfrak{e}) = 0 \leq kJ_\theta \; \forall \; \mathfrak{u}, \mathfrak{e} \in [0.1, 0.125] \cup \{0.2\}$. *So (37) holds.*
By the definition of Γ, $\Gamma\mathfrak{u}$ *is closed and bounded for all* $\mathfrak{u} \in \mathcal{H}^\star$.

(4) *We see that* $\mathfrak{p} = 0.2$ *is a B.P. point of* Γ, *since* $D(\mathfrak{p}, \Gamma\mathfrak{p}) = \rho_\theta(0.2, \{0.4\}) = 12.5 = dst(\mathcal{H}^\star, \mathcal{K}^\star)$.

5. Concluding Remarks

We summarize our conclusion as follows.

(1) We generalized the notion of b-G pseudo-distance J_b [29] by introducing an E.b-G pseudo-distance J_θ.

(2) We gave an example of E.b-G pseudo-distance J_θ which is not a b-G pseudo-distance J_b in the sense of [29].

(3) We proved B.P. point theorems for the multi-valued contraction mappings with respect to E.b-G pseudo-distance.

(4) Our results generalized some recent results in the literature from metric spaces and b-metric spaces to E.b-m spaces.

(5) By letting $\theta(\mathfrak{u}, \mathfrak{e}) = s$ for each $\mathfrak{u}, \mathfrak{e} \in \mathfrak{U}^\star$ where $s \geq 1$, Theorem 3 generalized the main result of [29] with the condition that $ks < 1$ (see Corollary 1).

(6) Theorem 4 is the generalization of the main result of A. Abkar [26] from metric space to E.b-m space.

(7) By letting $\theta(\mathfrak{u}, \mathfrak{e}) = 1$ for all $\mathfrak{u}, \mathfrak{e} \in \mathfrak{U}^\star$ and $\rho_\theta = d$, Theorem 4 generalized the main result of [26] (see Corollary 3).

6. Future Scope

The research motivation in this article for the readers is that several important F. point and B.P. point results can be obtained using our newly introduced generalized distance space.

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Appendix A

Since, according to Remark 2, every extended b-metric is an E.b-G pseudo-distance on $\mathfrak{U}^\star$, so in Theorem 4, the extended b-metric ρ_θ is an E.b-G pseudo-distance on $\mathfrak{U}^\star$. By replacing $J_\theta = \rho_\theta$ in Definitions 5 and 6, we obtain

$$D(\mathfrak{u}, \mathcal{K}^\star) = \inf_{\mathfrak{e}\in\mathcal{K}^\star} \rho_\theta(\mathfrak{u}, \mathfrak{e}),$$

$$dst(\mathcal{H}^\star, \mathcal{K}^\star) = \inf\{\rho_\theta(\mathfrak{u}, \mathfrak{e}) : \mathfrak{u} \in \mathcal{H}^\star,\ \mathfrak{e} \in \mathcal{K}^\star\},$$

$$\mathcal{H}_0 = \{\mathfrak{u} \in \mathcal{H}^\star : \rho_\theta(\mathfrak{u}, \mathfrak{e}) = dst(\mathcal{H}^\star, \mathcal{K}^\star) \text{ for some } \mathfrak{e} \in \mathcal{K}^\star\},$$

$$\mathcal{K}_0 = \{\mathfrak{e} \in \mathcal{K}^\star : \rho_\theta(\mathfrak{u}, \mathfrak{e}) = dst(\mathcal{H}^\star, \mathcal{K}^\star) \text{ for some } \mathfrak{u} \in \mathcal{H}^\star\}.$$

$$H^{\rho_\theta}(\mathcal{H}^\star, \mathcal{K}^\star) = \max\{\sup_{\mathfrak{u}\in\mathcal{H}^\star} \rho_\theta(\mathfrak{u}, \mathcal{K}^\star), \sup_{\mathfrak{e}\in\mathcal{K}^\star} \rho_\theta(\mathfrak{e}, \mathcal{H}^\star)\} \text{ for all } \mathcal{H}^\star, \mathcal{K}^\star \in CB(\mathfrak{U}^\star),$$

and the P^{J_θ}-property becomes P^{ρ_θ}-property of the pair $(\mathcal{H}^\star, \mathcal{K}^\star)$.

Let $\mathfrak{u}_0 \in \mathcal{H}_0, \mathfrak{e}_0 \in \Gamma\mathfrak{u}_0 \subseteq \mathcal{K}_0$. Then there exists $\mathfrak{u}_1 \in \mathcal{H}_0$ such that

$$\rho_\theta(\mathfrak{u}_1, \mathfrak{e}_0) = dst(\mathcal{H}^\star, \mathcal{K}^\star). \tag{A1}$$

Since $\mathfrak{u}_0, \mathfrak{u}_1 \in \mathcal{H}_0, \mathfrak{e}_0 \in \Gamma\mathfrak{u}_0$, and ρ_θ is an E.b-G pseudo-distance on $\mathfrak{U}^\star$, so by Lemma 2, there exists $\mathfrak{e}_1 \in \Gamma\mathfrak{u}_1 \subseteq \mathcal{K}_0$ such that

$$\rho_\theta(\mathfrak{e}_0, \mathfrak{e}_1) \le H^{\rho_\theta}(\Gamma\mathfrak{u}_0, \Gamma\mathfrak{u}_1) + k. \tag{A2}$$

Again, as $\mathfrak{e}_1 \in \mathcal{K}_0$, so there exists $\mathfrak{u}_2 \in \mathcal{H}_0$ such that

$$\rho_\theta(\mathfrak{u}_2, \mathfrak{e}_1) = dst(\mathcal{H}^\star, \mathcal{K}^\star). \tag{A3}$$

Now $\mathfrak{u}_1, \mathfrak{u}_2 \in \mathcal{H}_0, \mathfrak{e}_1 \in \Gamma\mathfrak{u}_1$, and ρ_θ is an E.b-G pseudo-distance on $\mathfrak{U}^\star$, so by Lemma 2, there exists $\mathfrak{e}_2 \in \Gamma\mathfrak{u}_2$ such that

$$\rho_\theta(\mathfrak{e}_1, \mathfrak{e}_2) \le H^{\rho_\theta}(\Gamma\mathfrak{u}_1, \Gamma\mathfrak{u}_2) + k^2. \tag{A4}$$

We continue the above process, then, by induction, we find $\{\mathfrak{u}_n : n \in \{0\} \cup \mathbb{N}\}$, and $\{\mathfrak{e}_n : n \in \{0\} \cup \mathbb{N}\}$ such that

(i) $\mathfrak{u}_n \in \mathcal{H}_0, \mathfrak{e}_n \in \mathcal{K}_0\ \forall\ n \in \{0\} \cup \mathbb{N}$;
(ii) $\mathfrak{e}_n \in \Gamma\mathfrak{u}_n\ \forall\ n \in \{0\} \cup \mathbb{N}$;
(iii) $\rho_\theta(\mathfrak{u}_n, \mathfrak{e}_{n-1}) = dst(\mathcal{H}^\star, \mathcal{K}^\star)\ \forall\ n \in \mathbb{N}$;
(iv) $\rho_\theta(\mathfrak{e}_{n-1}, \mathfrak{e}_n) \le H^{\rho_\theta}(\Gamma\mathfrak{u}_{n-1}, \Gamma\mathfrak{u}_n) + k^n\ \forall\ n \in \mathbb{N}$.

Now, for any $n \in \mathbb{N}$, we have $\rho_\theta(\mathfrak{u}_n, \mathfrak{e}_{n-1}) = dst(\mathcal{H}^\star, \mathcal{K}^\star)$ and $\rho_\theta(\mathfrak{u}_{n+1}, \mathfrak{e}_n) = dst(\mathcal{H}^\star, \mathcal{K}^\star)$. Since the pair $(\mathcal{H}^\star, \mathcal{K}^\star)$ has the P^{ρ_θ}-property, we deduce

$$\rho_\theta(\mathfrak{u}_n, \mathfrak{u}_{n+1}) = \rho_\theta(\mathfrak{e}_{n-1}, \mathfrak{e}_n),\ \forall\ n \in \mathbb{N}. \tag{A5}$$

Now for $\mathfrak{u} = \mathfrak{u}_n, \mathfrak{e} = \mathfrak{u}_{n+1}$, and $n \in \mathbb{N}$, by (27), we obtain

$$H^{\rho_\theta}(\Gamma\mathfrak{u}_n, \Gamma\mathfrak{u}_{n+1}) \le \frac{k}{s_{\mathfrak{u}_n}} \rho_\theta(\mathfrak{u}_n, \mathfrak{u}_{n+1})\ \forall\ n \in \{0\} \cup \mathbb{N}. \tag{A6}$$

Next, for all $n \in \mathbb{N}$, we have

$$
\begin{aligned}
\rho_\theta(\mathfrak{u}_n, \mathfrak{u}_{n+1}) &= \rho_\theta(\mathfrak{e}_{n-1}, \mathfrak{e}_n) \\
&\leq H^{\rho_\theta}(\Gamma\mathfrak{u}_{n-1}, \Gamma\mathfrak{u}_n) + k^n \\
&\leq \frac{k}{s_{\mathfrak{u}_{n-1}}}\rho_\theta(\mathfrak{u}_{n-1}, \mathfrak{u}_n) + k^n \\
&\leq k\rho_\theta(\mathfrak{u}_{n-1}, \mathfrak{u}_n) + k^n \\
&= k\rho_\theta(\mathfrak{e}_{n-2}, \mathfrak{e}_{n-1}) + k^n \\
&\leq k\left[H^{\rho_\theta}(\Gamma\mathfrak{u}_{n-2}, \Gamma\mathfrak{u}_{n-1}) + k^{n-1}\right] + k^n \\
&= kH^{\rho_\theta}(\Gamma\mathfrak{u}_{n-2}, \Gamma\mathfrak{u}_{n-1}) + 2k^n \\
&\leq \frac{k^2}{s_{\mathfrak{u}_{n-2}}}\rho_\theta(\mathfrak{u}_{n-2}, \mathfrak{u}_{n-1}) + 2k^n \\
&\leq k^2\rho_\theta(\mathfrak{u}_{n-2}, \mathfrak{u}_{n-1}) + 2k^n \\
&= k^2\rho_\theta(\mathfrak{e}_{n-3}, \mathfrak{e}_{n-2}) + 2k^n \\
&\leq k^2\left[H^{\rho_\theta}(\Gamma\mathfrak{u}_{n-3}, \Gamma\mathfrak{u}_{n-2}) + k^{n-2}\right] + 2k^n \\
&= k^2H^{\rho_\theta}(\Gamma\mathfrak{u}_{n-3}, \Gamma\mathfrak{u}_{n-2}) + 3k^n \\
&\leq \frac{k^3}{s_{\mathfrak{u}_{n-3}}}\rho_\theta(\mathfrak{u}_{n-3}, \mathfrak{u}_{n-2}) + 3k^n \\
&\leq k^3\rho_\theta(\mathfrak{u}_{n-3}, \mathfrak{u}_{n-2}) + 3k^n \\
&\leq \cdots \leq k^n\rho_\theta(\mathfrak{u}_0, \mathfrak{u}_1) + nk^n.
\end{aligned}
$$

Therefore, we obtain

$$
\rho_\theta(\mathfrak{u}_n, \mathfrak{u}_{n+1}) \leq \ k^n\rho_\theta(\mathfrak{u}_0, \mathfrak{u}_1) + nk^n \ \forall\ n \in \mathbb{N}.
$$

Now, for each $n > m$, we obtain

$$
\begin{aligned}
\rho_\theta(\mathfrak{u}_m, \mathfrak{u}_n) &\leq \theta(\mathfrak{u}_m, \mathfrak{u}_n)[\rho_\theta(\mathfrak{u}_m, \mathfrak{u}_{m+1}) + \rho_\theta(\mathfrak{u}_{m+1}, \mathfrak{u}_n)] \\
&\leq \theta(\mathfrak{u}_m, \mathfrak{u}_n)\rho_\theta(\mathfrak{u}_m, \mathfrak{u}_{m+1}) + \theta(\mathfrak{u}_m, \mathfrak{u}_n)\theta(\mathfrak{u}_{m+1}, \mathfrak{u}_n)\rho_\theta(\mathfrak{u}_{m+1}, \mathfrak{u}_{m+2}) \\
&\qquad + \theta(\mathfrak{u}_m, \mathfrak{u}_n)\theta(\mathfrak{u}_{m+1}, \mathfrak{u}_n)\rho_\theta(\mathfrak{u}_{m+2}, \mathfrak{u}_n) \\
&\leq \cdots \leq \sum_{i=m}^{n-1}\left(\prod_{j=m}^{i}\theta(\mathfrak{u}_j, \mathfrak{u}_n)\rho_\theta(\mathfrak{u}_i, \mathfrak{u}_{i+1})\right) \\
&\leq \sum_{i=m}^{n-1}\left(\prod_{j=m}^{i}\theta(\mathfrak{u}_j, \mathfrak{u}_n)\left(k^i\rho_\theta(\mathfrak{u}_0, \mathfrak{u}_1) + ik^i\right)\right) \\
&= \sum_{i=m}^{n-1}\left(\prod_{j=m}^{i}\theta(\mathfrak{u}_j, \mathfrak{u}_n)k^i\rho_\theta(\mathfrak{u}_0, \mathfrak{u}_1) + \prod_{j=m}^{i}\theta(\mathfrak{u}_j, \mathfrak{u}_n)ik^i\right) \\
&= \sum_{i=m}^{n-1}\left(\prod_{j=m}^{i}\theta(\mathfrak{u}_j, \mathfrak{u}_n)k^i\rho_\theta(\mathfrak{u}_0, \mathfrak{u}_1)\right) + \sum_{i=m}^{n-1}\left(\prod_{j=m}^{i}\theta(\mathfrak{u}_j, \mathfrak{u}_n)ik^i\right) \\
&\leq \sum_{i=m}^{n-1}\left(\prod_{j=1}^{i}\theta(\mathfrak{u}_j, \mathfrak{u}_n)k^i\rho_\theta(\mathfrak{u}_0, \mathfrak{u}_1)\right) + \sum_{i=m}^{n-1}\left(\prod_{j=1}^{i}\theta(\mathfrak{u}_j, \mathfrak{u}_n)ik^i\right) \\
&= \rho_\theta(\mathfrak{u}_0, \mathfrak{u}_1)\sum_{i=m}^{n-1}\left(\prod_{j=1}^{i}\theta(\mathfrak{u}_j, \mathfrak{u}_n)k^i\right) + \sum_{i=m}^{n-1}\left(\prod_{j=1}^{i}\theta(\mathfrak{u}_j, \mathfrak{u}_n)ik^i\right).
\end{aligned}
$$

Let $a_m = \prod_{j=1}^{m} \theta(\mathfrak{u}_j, \mathfrak{u}_n)k^m$, $S_m = \sum_{i=1}^{m} \left(\prod_{j=1}^{i} \theta(\mathfrak{u}_j, \mathfrak{u}_n)k^i \right)$ and $a'_m = \prod_{j=1}^{m} \theta(\mathfrak{u}_m, \mathfrak{u}_n)mk^m$, $S'_m = \sum_{i=1}^{m} \left(\prod_{j=1}^{i} \theta(\mathfrak{u}_j, \mathfrak{u}_n)ik^i \right)$. Then $\lim_{m\to\infty} \frac{a_{m+1}}{a_m} = \lim_{m,n\to\infty} \theta(\mathfrak{u}_{m+1}, \mathfrak{u}_n)k < 1$, and

$$\begin{aligned}
\lim_{m\to\infty} \frac{a'_{m+1}}{a'_m} &= \lim_{m,n\to\infty} \theta(\mathfrak{u}_{m+1}, \mathfrak{u}_n)\left(\frac{m+1}{m}\right)k \\
&= \lim_{m,n\to\infty} \theta(\mathfrak{u}_{m+1}, \mathfrak{u}_n)\left(1 + \frac{1}{m}\right)k \\
&= \lim_{m,n\to\infty} \theta(\mathfrak{u}_{m+1}, \mathfrak{u}_n)k + \lim_{m,n\to\infty} \theta(\mathfrak{u}_{m+1}, \mathfrak{u}_n)\frac{k}{m} \\
&< 1 + 0.
\end{aligned}$$

Since $\lim_{m,n\to\infty} \theta(\mathfrak{u}_{m+1}, \mathfrak{u}_n)$ is finite and $\lim_{m\to\infty} \frac{k}{m} = 0$, the series $\sum_{i=1}^{\infty} \left(\prod_{j=1}^{i} \theta(\mathfrak{u}_j, \mathfrak{u}_n)k^i \right)$ and $\sum_{i=1}^{\infty} \left(\prod_{j=1}^{i} \theta(\mathfrak{u}_j, \mathfrak{u}_n)ik^i \right)$ converge by ratio test for each $n \in \mathbb{N}$. For $n > m$, above inequality implies

$$\rho_\theta(\mathfrak{u}_m, \mathfrak{u}_n) \leq \rho_\theta(\mathfrak{u}_0, \mathfrak{u}_1)[S_{n-1} - S_m] + S'_{n-1} - S'_m.$$

Letting $m \to \infty$, we conclude

$$\lim_{m\to\infty} \sup_{n>m} \rho_\theta(\mathfrak{u}_m, \mathfrak{u}_n) = 0. \tag{A7}$$

From (A12) and (A13), we have

$$\lim_{m\longrightarrow\infty} \sup_{n>m} \rho_\theta(\mathfrak{e}_m, \mathfrak{e}_n) = 0.$$

Also $\lim_{n,m\to\infty} \theta(\mathfrak{u}_n, \mathfrak{u}_m) < \frac{1}{k}$, and $\lim_{n,m\to\infty} \theta(\mathfrak{e}_n, \mathfrak{e}_m) < \frac{1}{k}$. By Lemma 1, the sequence $\{\mathfrak{u}_n : n \in \{0\} \cup \mathbb{N}\}$ is Cauchy in $\mathcal{H}^\star$ and $\{\mathfrak{e}_n : n \in \{0\} \cup \mathbb{N}\}$ is Cauchy in $\mathcal{K}^\star$. But since the subsets $\mathcal{H}^\star$ and $\mathcal{K}^\star$ are closed in the complete space $(\mathfrak{U}^\star, \rho_\theta)$, there is some $\mathfrak{p}$ in $\mathcal{H}^\star$ and $\mathfrak{q}$ in $\mathcal{K}^\star$, such that $\mathfrak{u}_n \to \mathfrak{p}$, $\mathfrak{e}_n \to \mathfrak{q}$. Since $\mathfrak{e}_n \in \Gamma\mathfrak{u}_n$ for all $n \in \{0\} \cup \mathbb{N}$ and the multi-valued non-self mapping Γ is closed, $\mathfrak{q} \in \Gamma\mathfrak{p}$. Since $\rho_\theta(\mathfrak{u}_n, \mathfrak{e}_{n-1}) = dst(\mathcal{H}^\star, \mathcal{K}^\star)$ and ρ_θ is associated with $(\mathcal{H}^\star, \mathcal{K}^\star)$, because of the continuity of ρ_θ (we have chosen ρ_θ to be continuous throughout), we deduce that $\rho_\theta(\mathfrak{p}, \mathfrak{q}) = dst(\mathcal{H}^\star, \mathcal{K}^\star)$. Now,

$$dst(\mathcal{H}^\star, \mathcal{K}^\star) = \rho_\theta(\mathfrak{p}, \mathfrak{q}) \geq D(\mathfrak{p}, \Gamma\mathfrak{p}) \geq D(\mathfrak{p}, \mathcal{K}^\star) \geq dst(\mathcal{H}^\star, \mathcal{K}^\star),$$

Hence,

$$D(\mathfrak{p}, \Gamma\mathfrak{p}) = dst(\mathcal{H}^\star, \mathcal{K}^\star).$$

This completes the proof.

Appendix B

Let $\mathfrak{u}_0 \in \mathcal{H}_0$, $\mathfrak{e}_0 = \Gamma\mathfrak{u}_0 \subseteq \mathcal{K}_0$. Then there exists $\mathfrak{u}_1 \in \mathcal{H}_0$ such that

$$J_\theta(\mathfrak{u}_1, \mathfrak{e}_0) = dst(\mathcal{H}^\star, \mathcal{K}^\star). \tag{A8}$$

Since $\mathfrak{u}_0, \mathfrak{u}_1 \in \mathcal{H}_0$, for $\mathfrak{e}_0 = \Gamma\mathfrak{u}_0$, $\mathfrak{e}_1 = \Gamma\mathfrak{u}_1 \in \mathcal{K}_0$, for all $k > 0$, we have

$$J_\theta(\mathfrak{e}_0, \mathfrak{e}_1) \leq \rho_\theta(\Gamma\mathfrak{u}_0, \Gamma\mathfrak{u}_1) + k. \tag{A9}$$

Again, as $\mathfrak{e}_1 \in \mathcal{K}_0$, so there exists $\mathfrak{u}_2 \in \mathcal{H}_0$ such that

$$J_\theta(\mathfrak{u}_2, \mathfrak{e}_1) = dst(\mathcal{H}^\star, \mathcal{K}^\star). \quad \text{(A10)}$$

Now $\mathfrak{u}_1, \mathfrak{u}_2 \in \mathcal{H}_0$, for $\mathfrak{e}_1 = \Gamma\mathfrak{u}_1$, $\mathfrak{e}_2 = \Gamma\mathfrak{u}_2$, we have

$$J_\theta(\mathfrak{e}_1, \mathfrak{e}_2) \leq \rho_\theta(\Gamma\mathfrak{u}_1, \Gamma\mathfrak{u}_2) + k^2. \quad \text{(A11)}$$

We continue the above process, then by induction, we find $\{\mathfrak{u}_n : n \in \{0\} \cup \mathbb{N}\}$, and $\{\mathfrak{e}_n : n \in \{0\} \cup \mathbb{N}\}$ such that

(i) $\quad \mathfrak{u}_n \in \mathcal{H}_0, \mathfrak{e}_n \in \mathcal{K}_0\ \forall\ n \in \{0\} \cup \mathbb{N}$;
(ii) $\quad \mathfrak{e}_n = \Gamma\mathfrak{u}_n\ \forall\ n \in \{0\} \cup \mathbb{N}$;
(iii) $\quad J_\theta(\mathfrak{u}_n, \mathfrak{e}_{n-1}) = dst(\mathcal{H}^\star, \mathcal{K}^\star)\ \forall\ n \in \mathbb{N}$;
(iv) $\quad J_\theta(\mathfrak{e}_{n-1}, \mathfrak{e}_n) \leq \rho_\theta(\Gamma\mathfrak{u}_{n-1}, \Gamma\mathfrak{u}_n) + k^n\ \forall\ n \in \mathbb{N}$.

Now, for any $n \in \mathbb{N}$, we have $J_\theta(\mathfrak{u}_n, \mathfrak{e}_{n-1}) = dst(\mathcal{H}^\star, \mathcal{K}^\star)$ and $J_\theta(\mathfrak{u}_{n+1}, \mathfrak{e}_n) = dst(\mathcal{H}^\star, \mathcal{K}^\star)$. Since the pair $(\mathcal{H}^\star, \mathcal{K}^\star)$ has the P^{J_θ}-property, we deduce

$$J_\theta(\mathfrak{u}_n, \mathfrak{u}_{n+1}) = J_\theta(\mathfrak{e}_{n-1}, \mathfrak{e}_n),\ \forall\ n \in \mathbb{N}. \quad \text{(A12)}$$

Now, for $\mathfrak{u} = \mathfrak{u}_n, \mathfrak{e} = \mathfrak{u}_{n+1},$ and $n \in \mathbb{N}$, by (28), we obtain

$$J_\theta(\Gamma\mathfrak{u}_n, \Gamma\mathfrak{u}_{n+1}) \leq \frac{k}{s_{\mathfrak{u}_n}} J_\theta(\mathfrak{u}_n, \mathfrak{u}_{n+1})\ \forall\ n \in \{0\} \cup \mathbb{N}. \quad \text{(A13)}$$

Next, for all $n \in \mathbb{N}$, we have

$$\begin{aligned}
J_\theta(\mathfrak{u}_n, \mathfrak{u}_{n+1}) &= J_\theta(\mathfrak{e}_{n-1}, \mathfrak{e}_n) \\
&\leq \rho_\theta(\Gamma\mathfrak{u}_{n-1}, \Gamma\mathfrak{u}_n) + k^n \\
&\leq \frac{k}{s_{\mathfrak{u}_{n-1}}} J_\theta(\mathfrak{u}_{n-1}, \mathfrak{u}_n) + k^n \\
&\leq k J_\theta(\mathfrak{u}_{n-1}, \mathfrak{u}_n) + k^n \\
&= k J_\theta(\mathfrak{e}_{n-2}, \mathfrak{e}_{n-1}) + k^n \\
&\leq k\left[J_\theta(\Gamma\mathfrak{u}_{n-2}, \Gamma\mathfrak{u}_{n-1}) + k^{n-1}\right] + k^n \\
&= k J_\theta(\Gamma\mathfrak{u}_{n-2}, \Gamma\mathfrak{u}_{n-1}) + 2k^n \\
&\leq \frac{k^2}{s_{\mathfrak{u}_{n-2}}} J_\theta(\mathfrak{u}_{n-2}, \mathfrak{u}_{n-1}) + 2k^n \\
&\leq k^2 J_\theta(\mathfrak{u}_{n-2}, \mathfrak{u}_{n-1}) + 2k^n \\
&= k^2 J_\theta(\mathfrak{e}_{n-3}, \mathfrak{e}_{n-2}) + 2k^n \\
&\leq k^2\left[J_\theta(\Gamma\mathfrak{u}_{n-3}, \Gamma\mathfrak{u}_{n-2}) + k^{n-2}\right] + 2k^n \\
&= k^2 J_\theta(\Gamma\mathfrak{u}_{n-3}, \Gamma\mathfrak{u}_{n-2}) + 3k^n \\
&\leq \frac{k^3}{s_{\mathfrak{u}_{n-3}}} J_\theta(\mathfrak{u}_{n-3}, \mathfrak{u}_{n-2}) + 3k^n \\
&\leq k^3 J_\theta(\mathfrak{u}_{n-3}, \mathfrak{u}_{n-2}) + 3k^n \\
&\leq \cdots \leq k^n J_\theta(\mathfrak{u}_0, \mathfrak{u}_1) + nk^n.
\end{aligned}$$

So we obtain

$$J_\theta(\mathfrak{u}_n, \mathfrak{u}_{n+1}) \leq\ k^n J_\theta(\mathfrak{u}_0, \mathfrak{u}_1) + nk^n\ \forall\ n \in \mathbb{N}.$$

Now, for each $n > m$, we obtain

$$\begin{aligned}
J_\theta(\mathfrak{u}_m,\mathfrak{u}_n) &\le \theta(\mathfrak{u}_m,\mathfrak{u}_n)[J_\theta(\mathfrak{u}_m,\mathfrak{u}_{m+1}) + J_\theta(\mathfrak{u}_{m+1},\mathfrak{u}_n)] \\
&\le \theta(\mathfrak{u}_m,\mathfrak{u}_n)J_\theta(\mathfrak{u}_m,\mathfrak{u}_{m+1}) + \theta(\mathfrak{u}_m,\mathfrak{u}_n)\theta(\mathfrak{u}_{m+1},\mathfrak{u}_n)J_\theta(\mathfrak{u}_{m+1},\mathfrak{u}_{m+2}) \\
&\quad + \theta(\mathfrak{u}_m,\mathfrak{u}_n)\theta(\mathfrak{u}_{m+1},\mathfrak{u}_n)J_\theta(\mathfrak{u}_{m+2},\mathfrak{u}_n) \\
&\le \cdots \le \sum_{i=m}^{n-1}\left(\prod_{j=m}^{i}\theta(\mathfrak{u}_j,\mathfrak{u}_n)J_\theta(\mathfrak{u}_i,\mathfrak{u}_{i+1})\right) \\
&\le \sum_{i=m}^{n-1}\left(\prod_{j=m}^{i}\theta(\mathfrak{u}_j,\mathfrak{u}_n)(k^iJ_\theta(\mathfrak{u}_0,\mathfrak{u}_1) + ik^i)\right) \\
&= \sum_{i=m}^{n-1}\left(\prod_{j=m}^{i}\theta(\mathfrak{u}_j,\mathfrak{u}_n)k^iJ_\theta(\mathfrak{u}_0,\mathfrak{u}_1) + \prod_{j=m}^{i}\theta(\mathfrak{u}_j,\mathfrak{u}_n)ik^i\right) \\
&= \sum_{i=m}^{n-1}\left(\prod_{j=m}^{i}\theta(\mathfrak{u}_j,\mathfrak{u}_n)k^iJ_\theta(\mathfrak{u}_0,\mathfrak{u}_1)\right) + \sum_{i=m}^{n-1}\left(\prod_{j=m}^{i}\theta(\mathfrak{u}_j,\mathfrak{u}_n)ik^i\right) \\
&\le \sum_{i=m}^{n-1}\left(\prod_{j=1}^{i}\theta(\mathfrak{u}_j,\mathfrak{u}_n)k^iJ_\theta(\mathfrak{u}_0,\mathfrak{u}_1)\right) + \sum_{i=m}^{n-1}\left(\prod_{j=1}^{i}\theta(\mathfrak{u}_j,\mathfrak{u}_n)ik^i\right) \\
&= J_\theta(\mathfrak{u}_0,\mathfrak{u}_1)\sum_{i=m}^{n-1}\left(\prod_{j=1}^{i}\theta(\mathfrak{u}_j,\mathfrak{u}_n)k^i\right) + \sum_{i=m}^{n-1}\left(\prod_{j=1}^{i}\theta(\mathfrak{u}_j,\mathfrak{u}_n)ik^i\right).
\end{aligned}$$

Let $a_m = \prod_{j=1}^{m}\theta(\mathfrak{u}_j,\mathfrak{u}_n)k^m,\ S_m = \sum_{i=1}^{m}\left(\prod_{j=1}^{i}\theta(\mathfrak{u}_j,\mathfrak{u}_n)k^i\right)$ and $a'_m = \prod_{j=1}^{m}\theta(\mathfrak{u}_m,\mathfrak{u}_n)mk^m$, $S'_m = \sum_{i=1}^{m}\left(\prod_{j=1}^{i}\theta(\mathfrak{u}_j,\mathfrak{u}_n)ik^i\right)$. Then $\lim_{m\to\infty}\frac{a_{m+1}}{a_m} = \lim_{m,n\to\infty}\theta(\mathfrak{u}_{m+1},\mathfrak{u}_n)k < 1$, and

$$\begin{aligned}
\lim_{m\to\infty}\frac{a'_{m+1}}{a'_m} &= \lim_{m,n\to\infty}\theta(\mathfrak{u}_{m+1},\mathfrak{u}_n)\left(\frac{m+1}{m}\right)k \\
&= \lim_{m,n\to\infty}\theta(\mathfrak{u}_{m+1},\mathfrak{u}_n)\left(1+\frac{1}{m}\right)k \\
&= \lim_{m,n\to\infty}\theta(\mathfrak{u}_{m+1},\mathfrak{u}_n)k + \lim_{m,n\to\infty}\theta(\mathfrak{u}_{m+1},\mathfrak{u}_n)\frac{k}{m} \\
&< 1 + 0.
\end{aligned}$$

Since $\lim_{m,n\to\infty}\theta(\mathfrak{u}_{m+1},\mathfrak{u}_n)$ is finite and $\lim_{m\to\infty}\frac{k}{m} = 0$, the series $\sum_{i=1}^{\infty}\left(\prod_{j=1}^{i}\theta(\mathfrak{u}_j,\mathfrak{u}_n)k^i\right)$ and $\sum_{i=1}^{\infty}\left(\prod_{j=1}^{i}\theta(\mathfrak{u}_j,\mathfrak{u}_n)ik^i\right)$ converge by ratio test for each $n \in \mathbb{N}$. For $n > m$, above inequality implies

$$J_\theta(\mathfrak{u}_m,\mathfrak{u}_n) \le J_\theta(\mathfrak{u}_0,\mathfrak{u}_1)[S_{n-1} - S_m] + S'_{n-1} - S'_m.$$

Letting $m \to \infty$, we conclude

$$\lim_{m\to\infty}\sup_{n>m} J_\theta(\mathfrak{u}_m,\mathfrak{u}_n) = 0. \tag{A14}$$

From (A8) and (A9), we have

$$\lim_{m\longrightarrow\infty}\sup_{n>m} J_\theta(\mathfrak{e}_m,\mathfrak{e}_n) = 0.$$

Also $\lim_{n,m\to\infty}\theta(\mathfrak{u}_n,\mathfrak{u}_m) < \frac{1}{k}$, and $\lim_{n,m\to\infty}\theta(\mathfrak{e}_n,\mathfrak{e}_m) < \frac{1}{k}$. By Lemma 1, the sequence $\{\mathfrak{u}_n : n \in \{0\}\cup\mathbb{N}\}$ is Cauchy in $\mathcal{H}^\star$ and $\{\mathfrak{e}_n : n \in \{0\}\cup\mathbb{N}\}$ is Cauchy in $\mathcal{K}^\star$. But since the subsets $\mathcal{H}^\star$ and $\mathcal{K}^\star$ are closed in the complete space $(\mathfrak{U}^\star, J_\theta)$, there is some $\mathfrak{p}$ in $\mathcal{H}^\star$ and $\mathfrak{q}$ in $\mathcal{K}^\star$ such that $\mathfrak{u}_n \to \mathfrak{p},\ \mathfrak{e}_n \to \mathfrak{q}$ where $\mathfrak{e}_n = \Gamma\mathfrak{u}_n\ \forall\ n \in \mathbb{N}$, Since the contraction mapping $\Gamma : \mathcal{H}^\star \to \mathcal{K}^\star$ is continuous, so $\Gamma\mathfrak{u}_n \to \Gamma\mathfrak{p}$ which implies $\mathfrak{e}_n \to \Gamma\mathfrak{p}$. As the limit of a sequence in $(\mathfrak{U}^\star, \rho_\theta)$ is unique, $\mathfrak{q} = \Gamma\mathfrak{p}$. Since $J_\theta(\mathfrak{u}_n, \mathfrak{e}_{n-1}) = dst(\mathcal{H}^\star, \mathcal{K}^\star)$ and J_θ is associated with $(\mathcal{H}^\star, \mathcal{K}^\star)$, we deduce that $\rho_\theta(\mathfrak{p}, \mathfrak{q}) = dst(\mathcal{H}^\star, \mathcal{K}^\star)$. Now,

$$dst(\mathcal{H}^\star, \mathcal{K}^\star) = \rho_\theta(\mathfrak{p}, \mathfrak{q}) = \rho_\theta(\mathfrak{p}, \Gamma\mathfrak{p})$$

Hence,

$$\rho_\theta(\mathfrak{p}, \Gamma\mathfrak{p}) = dst(\mathcal{H}^\star, \mathcal{K}^\star).$$

This completes the proof.

Appendix C

Since, by Remark 2, every extended b-metric is an E.b-G pseudo-distance on $\mathfrak{U}^\star$, so in Theorem 6, the extended b-metric ρ_θ is an E.b-G pseudo-distance on $\mathfrak{U}^\star$. By replacing $J_\theta = \rho_\theta$ in Definitions 5 and 6, we obtain

$$dst(\mathcal{H}^\star, \mathcal{K}^\star) = \inf\{\rho_\theta(\mathfrak{u}, \mathfrak{e}) : \mathfrak{u} \in \mathcal{H}^\star,\ \mathfrak{e} \in \mathcal{K}^\star\},$$

$$\mathcal{H}_0 = \{\mathfrak{u} \ \in \mathcal{H}^\star : \rho_\theta(\mathfrak{u}, \mathfrak{e}) = dst(\mathcal{H}^\star, \mathcal{K}^\star) \text{ for some } \mathfrak{e} \ \in \mathcal{K}^\star\},$$

$$\mathcal{K}_0 = \{\mathfrak{e} \ \in \mathcal{K}^\star : \rho_\theta(\mathfrak{u}, \mathfrak{e}) = dst(\mathcal{H}^\star, \mathcal{K}^\star) \text{ for some } \mathfrak{u} \in \mathcal{H}^\star\},$$

and the P^{J_θ}-property becomes P^{ρ_θ}-property of the pair $(\mathcal{H}^\star, \mathcal{K}^\star)$.

Let $\mathfrak{u}_0 \in \mathcal{H}_0, \mathfrak{e}_0 = \Gamma\mathfrak{u}_0 \subseteq \mathcal{K}_0$. Then, there exists $\mathfrak{u}_1 \in \mathcal{H}_0$ such that

$$\rho_\theta(\mathfrak{u}_1, \mathfrak{e}_0) = dst(\mathcal{H}^\star, \mathcal{K}^\star). \tag{A15}$$

Since $\mathfrak{u}_0, \mathfrak{u}_1 \in \mathcal{H}_0$, for $\mathfrak{e}_0 = \Gamma\mathfrak{u}_0$, $\mathfrak{e}_1 = \Gamma\mathfrak{u}_1 \in \mathcal{K}_0$, for all $k > 0$, we have

$$\rho_\theta(\mathfrak{e}_0, \mathfrak{e}_1) \leq \rho_\theta(\Gamma\mathfrak{u}_0, \Gamma\mathfrak{u}_1) + k. \tag{A16}$$

Again, as $\mathfrak{e}_1 \in \mathcal{K}_0$, so there exists $\mathfrak{u}_2 \in \mathcal{H}_0$, such that

$$\rho_\theta(\mathfrak{u}_2, \mathfrak{e}_1) = dst(\mathcal{H}^\star, \mathcal{K}^\star). \tag{A17}$$

Now $\mathfrak{u}_1, \mathfrak{u}_2 \in \mathcal{H}_0$, for $\mathfrak{e}_1 = \Gamma\mathfrak{u}_1$, $\mathfrak{e}_2 = \Gamma\mathfrak{u}_2$, we have

$$\rho_\theta(\mathfrak{e}_1, \mathfrak{e}_2) \leq \rho_\theta(\Gamma\mathfrak{u}_1, \Gamma\mathfrak{u}_2) + k^2. \tag{A18}$$

We continue the above process; then, by induction, we find $\{\mathfrak{u}_n : n \in \{0\} \cup \mathbb{N}\}$, and $\{\mathfrak{e}_n : n \in \{0\} \cup \mathbb{N}\}$ such that

(i) $\mathfrak{u}_n \in \mathcal{H}_0, \mathfrak{e}_n \in \mathcal{K}_0\ \forall\ n \in \{0\} \cup \mathbb{N}$;
(ii) $\mathfrak{e}_n = \Gamma\mathfrak{u}_n\ \forall\ n \in \{0\} \cup \mathbb{N}$;
(iii) $\rho_\theta(\mathfrak{u}_n, \mathfrak{e}_{n-1}) = dst(\mathcal{H}^\star, \mathcal{K}^\star)\ \forall\ n \in \mathbb{N}$;
(iv) $\rho_\theta(\mathfrak{e}_{n-1}, \mathfrak{e}_n) \leq \rho_\theta(\Gamma\mathfrak{u}_{n-1}, \Gamma\mathfrak{u}_n) + k^n\ \forall\ n \in \mathbb{N}$.

Now, for any $n \in \mathbb{N}$, we have $J_\theta(\mathfrak{u}_n, \mathfrak{e}_{n-1}) = dst(\mathcal{H}^\star, \mathcal{K}^\star)$ and $\rho_\theta(\mathfrak{u}_{n+1}, \mathfrak{e}_n) = dst(\mathcal{H}^\star, \mathcal{K}^\star)$. Since the pair $(\mathcal{H}^\star, \mathcal{K}^\star)$ has the P^{ρ_θ}-property, we deduce

$$\rho_\theta(\mathfrak{u}_n, \mathfrak{u}_{n+1}) = \rho_\theta(\mathfrak{e}_{n-1}, \mathfrak{e}_n),\ \forall\ n \in \mathbb{N}. \tag{A19}$$

Now for $\mathfrak{u} = \mathfrak{u}_n, \mathfrak{e} = \mathfrak{u}_{n+1}$, and $n \in \mathbb{N}$, by (29), we obtain

$$\rho_\theta(\Gamma\mathfrak{u}_n, \Gamma\mathfrak{u}_{n+1}) \leq \frac{k}{s_{\mathfrak{u}_n}}\rho_\theta(\mathfrak{u}_n, \mathfrak{u}_{n+1})\ \forall\ n \in \{0\} \cup \mathbb{N}. \tag{A20}$$

Next, for all $n \in \mathbb{N}$, we have

$$\begin{aligned}
\rho_\theta(\mathfrak{u}_n, \mathfrak{u}_{n+1}) &= \rho_\theta(\mathfrak{e}_{n-1}, \mathfrak{e}_n) \\
&\leq \rho_\theta(\Gamma\mathfrak{u}_{n-1}, \Gamma\mathfrak{u}_n) + k^n \\
&\leq \frac{k}{s_{\mathfrak{u}_{n-1}}} \rho_\theta(\mathfrak{u}_{n-1}, \mathfrak{u}_n) + k^n \\
&\leq k\rho_\theta(\mathfrak{u}_{n-1}, \mathfrak{u}_n) + k^n \\
&= k\rho_\theta(\mathfrak{e}_{n-2}, \mathfrak{e}_{n-1}) + k^n \\
&\leq k\left[\rho_\theta(\Gamma\mathfrak{u}_{n-2}, \Gamma\mathfrak{u}_{n-1}) + k^{n-1}\right] + k^n \\
&= k\rho_\theta(\Gamma\mathfrak{u}_{n-2}, \Gamma\mathfrak{u}_{n-1}) + 2k^n \\
&\leq \frac{k^2}{s_{\mathfrak{u}_{n-2}}} \rho_\theta(\mathfrak{u}_{n-2}, \mathfrak{u}_{n-1}) + 2k^n \\
&\leq k^2\rho_\theta(\mathfrak{u}_{n-2}, \mathfrak{u}_{n-1}) + 2k^n \\
&= k^2\rho_\theta(\mathfrak{e}_{n-3}, \mathfrak{e}_{n-2}) + 2k^n \\
&\leq k^2\left[\rho_\theta(\Gamma\mathfrak{u}_{n-3}, \Gamma\mathfrak{u}_{n-2}) + k^{n-2}\right] + 2k^n \\
&= k^2\rho_\theta(\Gamma\mathfrak{u}_{n-3}, \Gamma\mathfrak{u}_{n-2}) + 3k^n \\
&\leq \frac{k^3}{s_{\mathfrak{u}_{n-3}}} \rho_\theta(\mathfrak{u}_{n-3}, \mathfrak{u}_{n-2}) + 3k^n \\
&\leq k^3\rho_\theta(\mathfrak{u}_{n-3}, \mathfrak{u}_{n-2}) + 3k^n \\
&\leq \cdots \leq k^n\rho_\theta(\mathfrak{u}_0, \mathfrak{u}_1) + nk^n.
\end{aligned}$$

So we obtain

$$\rho_\theta(\mathfrak{u}_n, \mathfrak{u}_{n+1}) \leq \ k^n\rho_\theta(\mathfrak{u}_0, \mathfrak{u}_1) + nk^n \ \forall \ n \in \mathbb{N}.$$

Now, for each $n > m$, we obtain

$$\begin{aligned}
\rho_\theta(\mathfrak{u}_m, \mathfrak{u}_n) &\leq \theta(\mathfrak{u}_m, \mathfrak{u}_n)[\rho_\theta(\mathfrak{u}_m, \mathfrak{u}_{m+1}) + \rho_\theta(\mathfrak{u}_{m+1}, \mathfrak{u}_n)] \\
&\leq \theta(\mathfrak{u}_m, \mathfrak{u}_n)\rho_\theta(\mathfrak{u}_m, \mathfrak{u}_{m+1}) + \theta(\mathfrak{u}_m, \mathfrak{u}_n)\theta(\mathfrak{u}_{m+1}, \mathfrak{u}_n)\rho_\theta(\mathfrak{u}_{m+1}, \mathfrak{u}_{m+2}) \\
&\quad\ + \theta(\mathfrak{u}_m, \mathfrak{u}_n)\theta(\mathfrak{u}_{m+1}, \mathfrak{u}_n)\rho_\theta(\mathfrak{u}_{m+2}, \mathfrak{u}_n) \\
&\leq \cdots \leq \sum_{i=m}^{n-1}\left(\prod_{j=m}^{i}\theta(\mathfrak{u}_j, \mathfrak{u}_n)\rho_\theta(\mathfrak{u}_i, \mathfrak{u}_{i+1})\right) \\
&\leq \sum_{i=m}^{n-1}\left(\prod_{j=m}^{i}\theta(\mathfrak{u}_j, \mathfrak{u}_n)\left(k^i\rho_\theta(\mathfrak{u}_0, \mathfrak{u}_1) + ik^i\right)\right) \\
&= \sum_{i=m}^{n-1}\left(\prod_{j=m}^{i}\theta(\mathfrak{u}_j, \mathfrak{u}_n)k^i\rho_\theta(\mathfrak{u}_0, \mathfrak{u}_1) + \prod_{j=m}^{i}\theta(\mathfrak{u}_j, \mathfrak{u}_n)ik^i\right) \\
&= \sum_{i=m}^{n-1}\left(\prod_{j=m}^{i}\theta(\mathfrak{u}_j, \mathfrak{u}_n)k^i\rho_\theta(\mathfrak{u}_0, \mathfrak{u}_1)\right) + \sum_{i=m}^{n-1}\left(\prod_{j=m}^{i}\theta(\mathfrak{u}_j, \mathfrak{u}_n)ik^i\right) \\
&\leq \sum_{i=m}^{n-1}\left(\prod_{j=1}^{i}\theta(\mathfrak{u}_j, \mathfrak{u}_n)k^i\rho_\theta(\mathfrak{u}_0, \mathfrak{u}_1)\right) + \sum_{i=m}^{n-1}\left(\prod_{j=1}^{i}\theta(\mathfrak{u}_j, \mathfrak{u}_n)ik^i\right) \\
&= \rho_\theta(\mathfrak{u}_0, \mathfrak{u}_1)\sum_{i=m}^{n-1}\left(\prod_{j=1}^{i}\theta(\mathfrak{u}_j, \mathfrak{u}_n)k^i\right) + \sum_{i=m}^{n-1}\left(\prod_{j=1}^{i}\theta(\mathfrak{u}_j, \mathfrak{u}_n)ik^i\right).
\end{aligned}$$

Let $a_m = \prod_{j=1}^{m} \theta(\mathfrak{u}_j, \mathfrak{u}_n)k^m$, $S_m = \sum_{i=1}^{m} \left(\prod_{j=1}^{i} \theta(\mathfrak{u}_j, \mathfrak{u}_n)k^i \right)$ and $a'_m = \prod_{j=1}^{m} \theta(\mathfrak{u}_m, \mathfrak{u}_n)mk^m$, $S'_m = \sum_{i=1}^{m} \left(\prod_{j=1}^{i} \theta(\mathfrak{u}_j, \mathfrak{u}_n)ik^i \right)$. Then $\lim_{m\to\infty} \frac{a_{m+1}}{a_m} = \lim_{m,n\to\infty} \theta(\mathfrak{u}_{m+1}, \mathfrak{u}_n)k < 1$, and

$$\begin{aligned}
\lim_{m\to\infty} \frac{a'_{m+1}}{a'_m} &= \lim_{m,n\to\infty} \theta(\mathfrak{u}_{m+1}, \mathfrak{u}_n)\left(\frac{m+1}{m}\right)k \\
&= \lim_{m,n\to\infty} \theta(\mathfrak{u}_{m+1}, \mathfrak{u}_n)\left(1 + \frac{1}{m}\right)k \\
&= \lim_{m,n\to\infty} \theta(\mathfrak{u}_{m+1}, \mathfrak{u}_n)k + \lim_{m,n\to\infty} \theta(\mathfrak{u}_{m+1}, \mathfrak{u}_n)\frac{k}{m} \\
&< 1 + 0.
\end{aligned}$$

Since $\lim_{m,n\to\infty} \theta(\mathfrak{u}_{m+1}, \mathfrak{u}_n)$ is finite and $\lim_{m\to\infty} \frac{k}{m} = 0$, the series $\sum_{i=1}^{\infty} \left(\prod_{j=1}^{i} \theta(\mathfrak{u}_j, \mathfrak{u}_n)k^i \right)$ and $\sum_{i=1}^{\infty} \left(\prod_{j=1}^{i} \theta(\mathfrak{u}_j, \mathfrak{u}_n)ik^i \right)$ converge by ratio test for each $n \in \mathbb{N}$. For $n > m$, above inequality implies

$$\rho_\theta(\mathfrak{u}_m, \mathfrak{u}_n) \le \rho_\theta(\mathfrak{u}_0, \mathfrak{u}_1)[S_{n-1} - S_m] + S'_{n-1} - S'_m.$$

Letting $m \to \infty$, we conclude

$$\lim_{m\to\infty} \sup_{n>m} \rho_\theta(\mathfrak{u}_m, \mathfrak{u}_n) = 0. \tag{A21}$$

From (A19) and (A20), we have

$$\lim_{m\longrightarrow\infty} \sup_{n>m} \rho_\theta(\mathfrak{e}_m, \mathfrak{e}_n) = 0.$$

Also $\lim_{n,m\to\infty} \theta(\mathfrak{u}_n, \mathfrak{u}_m) < \frac{1}{k}$, and $\lim_{n,m\to\infty} \theta(\mathfrak{e}_n, \mathfrak{e}_m) < \frac{1}{k}$. By Lemma 1, the sequence $\{\mathfrak{u}_n : n \in \{0\} \cup \mathbb{N}\}$ is Cauchy in $\mathcal{H}^\star$ and $\{\mathfrak{e}_n : n \in \{0\} \cup \mathbb{N}\}$ is Cauchy in $\mathcal{K}^\star$.

But since the subsets $\mathcal{H}^\star$ and $\mathcal{K}^\star$ are closed in the complete space $(\mathfrak{U}^\star, J_\theta)$, there is some $\mathfrak{p}$ in $\mathcal{H}^\star$ and $\mathfrak{q}$ in $\mathcal{K}^\star$ such that $\mathfrak{u}_n \to \mathfrak{p}$, $\mathfrak{e}_n \to \mathfrak{q}$ where $\mathfrak{e}_n = \Gamma\mathfrak{u}_n$ $\forall\ n \in \mathbb{N}$, Since the contraction mapping $\Gamma : \mathcal{H}^\star \to \mathcal{K}^\star$ is continuous, so $\Gamma\mathfrak{u}_n \to \Gamma\mathfrak{p}$ which implies $\mathfrak{e}_n \to \Gamma\mathfrak{p}$. As the limit of a sequence in $(\mathfrak{U}^\star, \rho_\theta)$ is unique, $\mathfrak{q} = \Gamma\mathfrak{p}$. Since $\rho_\theta(\mathfrak{u}_n, \mathfrak{e}_{n-1}) = dst(\mathcal{H}^\star, \mathcal{K}^\star)$ and ρ_θ is associated with $(\mathcal{H}^\star, \mathcal{K}^\star)$, because of the continuity of ρ_θ (we have chosen ρ_θ to be continuous throughout), we deduce that $\rho_\theta(\mathfrak{p}, \mathfrak{q}) = dst(\mathcal{H}^\star, \mathcal{K}^\star)$. Now,

$$dst(\mathcal{H}^\star, \mathcal{K}^\star) = \rho_\theta(\mathfrak{p}, \mathfrak{q}) = \rho_\theta(\mathfrak{p}, \Gamma\mathfrak{p})$$

Hence,

$$\rho_\theta(\mathfrak{p}, \Gamma\mathfrak{p}) = dst(\mathcal{H}^\star, \mathcal{K}^\star).$$

This completes the proof.

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