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Coupled Fixed Point Theory in Subordinate Semimetric Spaces

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Abstract: The aim of this paper is to study the coupled fixed point of a class of mixed monotone operators in the setting of a subordinate semimetric space. Using the symmetry between the subordinate semimetric space and a JS-space, we generalize the results of Senapati and Dey on JS-spaces. In this paper, we obtain some coupled fixed point results and support them with some examples.

Keywords: subordinate semimetric space; coupled fixed point; Ψ_+ space; mixed monotone operator

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1. Introduction and Preliminaries

One of the most important tools in nonlinear functional analysis is fixed point theory. It is very well known that most nonlinear analysis problems can be treated as fixed point problems. Banach proposed that each contraction on a complete metric space possesses a unique fixed point. In [1], J. Villa-Morales introduced the concept of subordinate semimetric spaces. A subordinate semimetric space is an extension of the concept of the RS-space introduced by Rolda'n and Shahzad in [2]. Also, the notions of Jleli and Samet's metric space and Branciari's generalized metric space are special cases of an RS-space. The purpose of this article is to study the existence of coupled fixed points (CFPs) on complete subordinate semimetric spaces. We also aim to provide some applications and examples to illustrate our results. In this article, we operate on the set of extended real numbers using standard arithmetic operations, $\bar{\mathbb{R}} = \mathbb{R} \cup \{\infty, -\infty\}$, and the notations have their regular meanings. Let Γ be a nonempty set. We begin with an extension of the definition of a semimetric space.

Definition 1 ([1])**.** *Let Γ be a nonempty set. A semimetric space is a pair (Γ, Ψ) where $\Psi : \Gamma^2 \to [0, \infty]$ is a function that meets the following conditions:*
(D1) For each $(\omega, \nu) \in \Gamma^2$, if $\Psi(\omega, \nu) = 0$, then $\omega = \nu$;
(D2) For each $(\omega, \nu) \in \Gamma^2$, $\Psi(\omega, \nu) = \Psi(\nu, \omega)$.

We will use this notion to define several fundamental topological concepts.

Definition 2 ([1])**.** *Assume (Γ, Ψ) is a semimetric space. Let $\omega \in \Gamma$, and let $\{\omega_n\}$ be a sequence in Γ. Then, we have the following:*
(i) $\{\omega_n\}$ is called a convergent sequence to ω if $\lim_{n\to\infty} \Psi(\omega, \omega_n) = 0$.
(ii) $\{\omega_n\}$ is called a Cauchy sequence if $\lim_{n,m\to\infty} \Psi(\omega_n, \omega_m) = 0$.
(iii) The pair (Γ, Ψ) is called a complete semimetric space if each Cauchy sequence in Γ is convergent.

Our approach exhibits symmetry with the generalized metric space concept that Jleli and Samet established in their work [3]. Rolda'n and Shahzad [2] promptly generalized this concept in the manner described below.

Definition 3 ([1])**.** *A semimetric space* (Γ, Ψ) *is an RS-space if there exists* $c > 0$ *such that if* $\omega, \nu \in \Gamma$ *are two points and* $\{\omega_n\}$ *is an infinite Cauchy sequence and* $\lim_{n\to\infty} \Psi(\omega_n, \omega) = 0$*, then*

$$\Psi(\omega, \nu) \leq c \limsup_{n\to\infty} \Psi(\omega_n, \nu).$$

The special cases of the RS-space concept include the concepts of quasimetric spaces, modular spaces, generalized metric spaces, and Branciari's generalized metric spaces (see [2,3]).

Definition 4 ([1])**.** *We say that a semimetric space* (Γ, Ψ) *is a subordinate if there exists a function* $\xi : [0, \infty] \to [0, \infty]$ *with the following:*

(SO1) ξ *is non-decreasing and* $\lim_{\omega\to 0} \xi(\omega) = 0$*;*

(SO2) For every $(\omega, \nu) \in \Gamma^2$*, with* $\omega \neq \nu$*, and when* $\{\omega_n\}$ *is an infinite Cauchy sequence in* Γ *such that* $\{\omega_n\}$ *is convergent to* ω*, we have*

$$\Psi(\omega, \nu) \leq \xi(\limsup_{n\to\infty} \Psi(\omega_n, \nu)).$$

Then, the pair (Γ, Ψ) *is said to be subordinate to* ξ *or that* (Γ, Ψ) *is a subordinate semimetric space.*

Remark 1. *Note that each RS-space is a subordinate semimetric space (if we take* $\xi(\omega) = cx$*), but the converse is not true. Examples 2, 3, and 5 of [1] are subordinate semimetric spaces but they are not RS-spaces.*

The next proposition proves the uniqueness of the limit of a convergent sequence in a subordinate semimetric space, which is necessary for our main results.

Proposition 1. *Let* (ω_n) *be an infinite Cauchy sequence in a subordinate semimetric space* (Γ, Ψ)*. Suppose that* (ω_n) *converges to* ω *and* λ*. Then,* $\omega = \lambda$*.*

Proof. Suppose that $\omega \neq \lambda$. Then, by condition $(SO2)$, there is a function $\xi : [0, \infty] \to [0, \infty]$ where ξ is non-decreasing, $\lim_{\omega\to 0} \xi(\omega) = 0$, and

$$\begin{aligned} \Psi(\omega, \lambda) &\leq \xi(\limsup_{n\to\infty} \Psi(\omega_n, \lambda)) \\ &= \xi(0) = 0. \end{aligned}$$

Thus, $\omega = \lambda$. □

Proposition 2. *Let* (ω_n) *be an infinite Cauchy sequence in a subordinate semimetric space* (Γ, Ψ) *that converges to* $\omega \in \Gamma$*. Then,* $\Psi(\omega, \omega) = 0$*.*

Proof. By condition $(SO2)$, there is a function $\xi : [0, \infty] \to [0, \infty]$ where ξ is non-decreasing, $\lim_{\omega\to 0} \xi(\omega) = 0$, and

$$\Psi(\omega, \omega) \leq \xi(\limsup_{n\to\infty} \Psi(\omega_n, \omega)) = \xi(0) = 0.$$

□

In the context of partially ordered (PO) metric spaces, Bhaskar and Lakshmikantham [4] introduced the notion of coupled fixed points (CFPs) as follows:

Definition 5 ([4])**.** *An element* $(\omega, \nu) \in \Gamma^2$ *is said to be a CFP of the function* $G : \Gamma^2 \to \Gamma$ *if* $\omega = G(\omega, \nu)$ *and* $\nu = G(\nu, \omega)$*.*

Additionally, they presented the notion of the MM operator as follows.

Definition 6 ([4]). *Assume $(\Gamma, \leq)$ is a PO set and $G : \Gamma^2 \to \Gamma$ is a function. Then, we say that G has the MM property if the following hold:*

(MM-1) $\omega_1 \leq \omega_2 \Rightarrow G(\omega_1, \nu) \leq G(\omega_2, \nu)$ for all $\omega_1, \omega_2, \nu \in \Gamma$;
(MM-2) $\nu_1 \leq \nu_2 \Rightarrow G(\omega, \nu_1) \geq G(\omega, \nu_2)$ for all $\omega, \nu_1, \nu_2 \in \Gamma$.

Given this notion, the authors of [4] established the next theorem, which shows the existence of the CFP of an operator with the MM property in the setting of a complete PO metric space.

Theorem 1 ([4]). *Let (Γ, d) be a complete PO metric space. Let $k \in (0, 1)$ and suppose $G : \Gamma^2 \to \Gamma$ is an MM operator with the following property:*

$$d(G(\omega, \nu), G(\lambda, \mu)) \leq \frac{k}{2}\{d(\omega, \lambda) + d(\nu, \mu)\} \text{ for all } \omega \geq \lambda; \nu \leq \mu. \tag{1}$$

Also, consider that there exist $\omega_0, \nu_0 \in \Gamma$ with $\omega_0 \leq G(\omega_0, \nu_0)$ and $\nu_0 \geq G(\nu_0, \omega_0)$. If (I) G is continuous or (II) Γ has the following properties:

(i) If a non-decreasing sequence (ω_n) is convergent to ω, then $\omega_n \leq \omega$ for every $n \in \mathbb{N}$;
(ii) If a non-increasing sequence (ν_n) is convergent to ν, then $\nu_n \geq \nu$ for all $n \in \mathbb{N}$;

then there exist $\omega, \nu \in \Gamma$ such that $\omega = G(\omega, \nu)$ and $\nu = G(\nu, \omega)$.

The contraction condition (1) was then generalized by Berinde [5] in 2011 as follows:

$$d(G(\omega, \nu), G(\lambda, \mu)) + d(G(\nu, \omega), G(\mu, \lambda)) \leq \ k\{d(\omega, \lambda) + d(\nu, \mu)\} \text{ for all } \omega \geq \lambda; \nu \leq \mu \tag{2}$$

This condition proved a CFP for an MM operator on partially ordered complete metric spaces.

Senapati and Dey in [6] improved and extended Berinde's CFP findings in [5] using the condition of contraction (2) for an MM operator on partially ordered complete JS-metric spaces.

In this work, motivated by the concepts of subordinate semimetric spaces, we extended and improved the CFP findings of Senapati and Dey [6] due to the condition of contraction (2) for an MM operator in PO complete subordinate semimetric spaces. In order to support our main finding, we constructed some examples.

2. Main Results

We will first provide some notions related to the structure before introducing our main results.

Let (Γ, Ψ) be a partially ordered subordinate semimetric space to some function ξ. Consider Γ^2 and define a partial order on Γ^2 as $(\omega, \nu) \leq (\mu, \lambda) \Leftrightarrow \omega \geq \mu\,; \nu \leq \lambda$. Define a distance function $\Psi_+ : \Gamma^2 \times \Gamma^2 \to [0, \infty]$ as

$$\Psi_+((\omega, \nu), (\mu, \lambda)) = \Psi(\omega, \mu) + \Psi(\nu, \lambda).$$

Then, (Γ^2, Ψ_+) is a partially ordered subordinate semimetric space to $\xi' = \max\{2\xi, \xi + \xi_1 \mid \xi_1(t) = t\}$.

To see this, note the following:

(D_1) $\Psi_+((\omega, \nu), (\lambda, \mu)) = 0$. This implies that $\Psi(\omega, \lambda) + \Psi(\nu, \mu) = 0$. This is only possible if both $\Psi(\omega, \lambda) = 0$ and $\Psi(\nu, \mu) = 0$, i.e., $\omega = \lambda$ and $\nu = \mu$. Hence,

$$\Psi_+((\omega, \nu), ((\lambda, \mu))) = 0 \Rightarrow (\omega, \nu) = (\lambda, \mu) \text{ for all } (\omega, \nu), (\lambda, \mu) \in \Gamma^2.$$

(D_2) Clearly, $\Psi_+((\omega, \nu), (\lambda, \mu)) = \Psi_+((\lambda, \mu), (\omega, \nu))$ for all $(\omega, \nu), (\lambda, \mu) \in \Gamma^2$.

Then, (Γ^2, Ψ_+) is a semimetric space.

Next, let $\{(\omega_n, \nu_n)\}_{n\in\mathbb{N}}$ be an infinite Cauchy sequence in Γ^2 that is convergent to (ω, ν). There are two cases.

Case (1): Both sequences, $\{\omega_n\}_{n\in\mathbb{N}}$ and $\{\nu_n\}_{n\in\mathbb{N}}$, are infinite Cauchy sequences that converge to ω and ν, respectively. Then,

$$\begin{aligned}
\Psi_+((\omega,\nu),(\mu,\lambda)) &= \Psi(G(\omega,\mu)) + \Psi(\nu,\lambda)) \\
&\leq \xi\left(\limsup_{n\to\infty} \Psi(\omega_n,\mu)\right) + \xi\left(\limsup_{n\to\infty} \Psi(\nu_n,\lambda)\right) \\
&\leq \xi\left(\limsup_{n\to\infty} \Psi(\omega_n,\mu) + \limsup_{n\to\infty} \Psi(\nu_n,\lambda)\right) + \left(\limsup_{n\to\infty} \Psi(\omega_n,\mu) + \limsup_{n\to\infty} \Psi(\nu_n,\lambda)\right) \\
&\leq \xi\left(\limsup_{n\to\infty}(\Psi(\omega_n,\mu) + \Psi(\nu_n,\lambda))\right) + \xi\left(\limsup_{n\to\infty}(\Psi(\omega_n,\mu) + \Psi(\nu_n,\lambda))\right) \\
&= 2\xi\left(\limsup_{n\to\infty} \Psi(\omega_n,\mu) + \limsup_{n\to\infty} \Psi(\nu_n,\lambda)\right).
\end{aligned}$$

Case (2): One of $\{\omega_n\}_{n\in\mathbb{N}}$ and $\{\nu_n\}_{n\in\mathbb{N}}$ is finite (say $\{\nu_n\}_{n\in\mathbb{N}}$). Then, $\nu_n = \nu$ for all $n \geq n_0$, for some $n_0 \in \mathbb{N}$. Let $\xi_1(t) = t$.

$$\begin{aligned}
\Psi_+((\omega,\nu),(\mu,\lambda)) &= \Psi(G(\omega,\mu)) + \Psi(\nu,\lambda)) \\
&\leq \xi\left(\limsup_{n\to\infty} \Psi(\omega_n,\mu)\right) + \limsup_{n\to\infty} \Psi(\nu_n,\lambda) \\
&\leq \xi\left(\limsup_{n\to\infty}(\Psi(\omega_n,\mu) + \Psi(\nu_n,\lambda))\right) + \left(\limsup_{n\to\infty}(\Psi(\omega_n,\mu) + \Psi(\nu_n,\lambda))\right) \\
&= (\xi + \xi_1)\left(\limsup_{n\to\infty}(\Psi(\omega_n,\mu) + \Psi(\nu_n,\lambda))\right).
\end{aligned}$$

Let $\xi'(t) = \max\{2\xi(t), (\xi + \xi_1)(t)\}$.

From the above, we see that there exists a function $\xi' : [0,\infty] \to [0,\infty]$ with the following conditions:

(SO1) ξ' is non-decreasing and $\lim_{\omega\to 0} \xi'(\omega) = 0$;

(SO2) For every $((\omega,\nu),(\mu,\lambda)) \in \Gamma^2 \times \Gamma^2$, with $(\omega,\nu) \neq (\mu,\lambda)$, and when $\{(\omega_n,\nu_n)\}$ is an infinite Cauchy sequence in Γ^2 such that $\{(\omega_n,\nu_n)\}$ is convergent to (ω,ν), we have

$$\Psi_+((\omega,\nu),(\mu,\lambda)) \leq \xi'(\limsup_{n\to\infty} \Psi_+((\omega_n,\nu_n),(\mu,\lambda))).$$

Thus, (Γ^2, Ψ_+) is a partially ordered subordinate semimetric space to $\xi' = \max\{2\xi, \xi + \xi_1 \mid \xi_1(t) = t\}$.

In a similar fashion, we define a distance function on each n-tuple set Γ^n for each $n \geq 2$.

Thus, we define the function $\Psi_{\mathrm{m}} : \Gamma^2 \to \Gamma$ as

$$\Psi_{\mathrm{m}}((\omega,\nu),(\lambda,\mu)) = \max\{\Psi(\omega,\lambda), \Psi(\nu,\mu)\}.$$

It is easy to check that Ψ_{m} meets the axioms of a subordinate semimetric space.

Furthermore, $(\Gamma^2, \Psi_{\mathrm{m}})$ is a Ψ_{m}-subordinate semimetric space. Proceeding in this way, we may establish an n-tuple Ψ_{m}-subordinate semimetric space for each $n \geq 2$.

The following proposition will be necessary in order to state our main results.

Remark 2. *By Proposition (1), the limit of a Cauchy convergent sequence is unique in the subordinate semimetric space (Γ^2, Ψ_+), i.e, if $(\sigma_n) = (\omega_n, \nu_n)$ is an infinite Cauchy sequence in (Γ^2, Ψ_+), with (σ_n), Ψ_+ converges to $\omega^* = (\omega, \nu)$, and $\lambda^* = (\lambda, \mu)$ and then $\omega^* = \lambda^*$.*

By Proposition (2), we may derive the next argument.

Proposition 3. *Assume* (σ_n) *is an infinite Cauchy sequence in* (Γ^2, Ψ_+) *that converges to* $(\omega, \nu) \in \Gamma^2$*, where* $\sigma_n = (\omega_n, \nu_n)$*. Then,* $\Psi_+((\omega, \nu), (\omega, \nu)) = 0$.

If (Γ, Ψ) is a complete subordinate semimetric space, then it is easy to show that (Γ^2, Ψ_+) and $(\Gamma^2, \Psi_{\mathrm{m}})$ are complete as well.

Let $(\omega, \nu) \in \Gamma^2$, and let there be a function of $G : \Gamma^2 \to \Gamma$ that has an MM operator. We have defined

$$\delta_G(\Psi, (\omega, \nu)) = \sup\{\Psi(G^i(\omega, \nu), G^j(\omega, \nu)) : i, j \in \mathbb{N}\}$$

and

$$\delta_G(\Psi, (\nu, \omega)) = \sup\{\Psi(G^i(\nu, \omega), G^j(\nu, \omega)) : i, j \in \mathbb{N}\},$$

where

$$G^2(\omega, \nu) = G(G(\omega, \nu), G(\nu, \omega)); G^i(\omega, \nu) = G(G^{i-1}(\omega, \nu), G^{i-1}(\nu, \omega)).$$

Remember that the partial order $' \leq'$ on Γ^2 is defined in the following manner:

$$(\lambda, \mu) \leq (\omega, \nu) \Leftrightarrow \lambda \leq \omega, \mu \geq \nu$$

for all $\omega, \nu, \lambda, \mu \in \Gamma$.

CFP Results

Throughout this part, we generalize the works of Senapati and Dey [6], improving the results of Berinde [5].

Let (Γ, Ψ) be a PO subordinate semimetric space to some function ξ.

Let (Γ^2, Ψ_+) be the PO complete subordinate semimetric space (induced by (Γ, Ψ)).

Let $G : \Gamma^2 \to \Gamma$ be an MM operator.

The contraction condition (2) is written as follows:

$$\Psi(G(\omega, \nu), G(\lambda, \mu)) + \Psi(G(\nu, \omega), G(\mu, \lambda)) \leq k\{\Psi(\omega, \lambda) + \Psi(\nu, \mu)\} \tag{3}$$

for all $\omega \geq \lambda, \nu \leq \mu$, and $k \in (0, 1)$.

Now, we define an operator $T_G : \Gamma^2 \to \Gamma^2$ as

$$T_G(\omega, \nu) = (G(\omega, \nu), G(\nu, \omega)) \tag{4}$$

for all $(\omega, \nu) \in \Gamma^2$. Thus, the contraction condition (3) is presented as

$$\Psi_+(T_G(\Sigma), T_G(\Delta)) \leq k\Psi_+(\Sigma, \Delta) \tag{5}$$

where $\Sigma = (\omega, \nu), \Delta = (\lambda, \mu) \in \Gamma^2$ with $\omega \geq \lambda, \nu \leq \mu$, and $k \in (0, 1)$.

Remark 3. *Clearly, from the above, the CFP theorem for G simplifies to the common fixed point theorem for* T_G*, as* T_G *has a fixed point if and only if G has a CFP.*

Let $\sigma_0 = (\omega_0, \nu_0) \in \Gamma^2$. We how define

$$\delta(\Psi_+, T_G, \sigma_0) = \sup\{\Psi_+({T_G}^i(\sigma_0), {T_G}^j(\sigma_0)) : i, j \in \mathbb{N}\}.$$

The extended version of Senapati and Dey's results in [6] is shown in the following results.

Theorem 2. *Consider that* $G : \Gamma^2 \to \Gamma$ *is a mapping with an MM operator on a PO complete* Ψ_+*-subordinate semimetric space* (Γ^2, Ψ_+)*. Assume that for all* $\omega \geq \lambda$ *and* $\nu \leq \mu$*,*

G meets the condition of contraction (3). If there exists $\sigma_0 = (\omega_0, \nu_0) \in \Gamma^2$ with the following conditions:

(i) $\omega_0 \leq G(\omega_0, \nu_0)$ *and* $\nu_0 \geq G(\nu_0, \omega_0)$ *or* $\omega_0 \geq G(\omega_0, \nu_0)$ *and* $\nu_0 \leq G(\nu_0, \omega_0)$*;*
(ii) $\delta_G(\Psi, (\omega_0, \nu_0)) < \infty$ *and* $\delta_G(\Psi, (\nu_0, \omega_0)) < \infty$*;*

then there exists a CFP $\tilde{\sigma} = (\tilde{\omega}, \tilde{\nu}) \in \Gamma^2$ *of G, i.e.,* $\tilde{\omega} = G(\tilde{\omega}, \tilde{\nu})$*;* $\tilde{\nu} = G(\tilde{\nu}, \tilde{\omega})$*.*

Proof. By the hypothesis, assume that there exists $\sigma_0 = (\omega_0, \nu_0) \in \Gamma^2$ with $\omega_0 \leq G(\omega_0, \nu_0)$ and $\nu_0 \geq G(\nu_0, \omega_0)$.

Let $\omega_1 = G(\omega_0, \nu_0)$ and $\nu_1 = G(\nu_0, \omega_0)$, and we denote the following:

$$G^2(\omega_0, \nu_0) = G(G(\omega_0, \nu_0), G(\nu_0, \omega_0)) = G(\omega_1, \nu_1) = \omega_2;$$
$$G^2(\nu_0, \omega_0) = G(G(\nu_0, \omega_0), G(\omega_0, \nu_0)) = G(\omega_1, \omega_1) = \nu_2.$$

In a similar fashion, since G is an MM operator, we obtain

$$G^n(\omega_0, \nu_0) = G(G^{n-1}(\omega_0, \nu_0), G^{n-1}(\nu_0, \omega_0)) = \omega_n;$$
$$G^n(\nu_0, \omega_0) = G(G^{n-1}(\nu_0, \omega_0), G^{n-1}(\omega_0, \nu_0)) = \nu_n.$$

Throughout Remark 3, to establish the presence of a CFP of G, it is enough to prove the presence of a fixed point of T_G provided by Equation (4). To demonstrate this, let us assume

$$\sigma_1 = (\omega_1, \nu_1) = (G(\omega_0, \nu_0), G(\nu_0, \omega_0)) = T_G(\omega_0, \nu_0) = T_G(\sigma_0).$$

and

$$\sigma_2 = (\omega_2, \nu_2) = (G^2(\omega_0, \nu_0), G^2(\nu_0, \omega_0)) = (G(\omega_1, \nu_1), G(\nu_1, \omega_1)) = T_G(\sigma_1) = T_G^2(\sigma_0).$$

Proceeding in this way, we obtain

$$\sigma_n = (\omega_n, \nu_n) = (G^n(\omega_0, \nu_0), G^n(\nu_0, \omega_0)) = ... = T_G^n(\sigma_0) \text{ for all } n \in \mathbb{N}.$$

Thus, $\{\sigma_n\}$ is a Picard sequence that has the initial approximation σ_0. Also, since G is an MM operator, one can easily check that for any $n \geq 0$, $\omega_n \leq \omega_{n+1}$ and $\nu_n \geq \nu_{n+1}$. Thus $\sigma_n \leq \sigma_{n+1}$, i.e., $\{\sigma_n\}$ is a non-decreasing sequence.

Now, we can show that $\{\sigma_n\}$ is a Cauchy sequence due to the fact that G meets the condition of contraction (3), for each $n \geq 0$ and $i \leq j$. Therefore, we have the following:

$$\begin{aligned}
&\Psi(G^{n+i}(\omega_0, \nu_0), G^{n+j}(\omega_0, \nu_0)) + \Psi(G^{n+i}(\nu_0, \omega_0), G^{n+j}(\nu_0, \omega_0)) \\
&\leq k[\Psi(G^{n+i-1}(\omega_0, \nu_0), G^{n+j-1}(\omega_0, \nu_0)) + \Psi(G^{n+i-1}(\nu_0, \omega_0), G^{n+j-1}(\nu_0, \omega_0))] \\
&\Rightarrow \Psi_+({T_G}^{n+i}(\sigma_0), {T_G}^{n+j}(\sigma_0)) \\
&\leq k\Psi_+({T_G}^{n-1+i}(\sigma_0, {T_G}^{n-1+j}(\sigma_0))) && \text{[by (5)]} \\
&\Rightarrow \delta(\Psi_+, T_G, T_G^n(\sigma_0) \\
&\leq k\delta(\Psi_+, T_G, T_G^{n-1}(\sigma_0)).
\end{aligned}$$

This holds for each $n \in \mathbb{N}$ such that for each $i \leq j$, we obtain

$$\begin{aligned}
\Psi_+({T_G}^{n+i}(\sigma_0), {T_G}^{n+j}(\sigma_0)) &\leq k\delta(\Psi_+, T_G, {T_G}^{n-1}(\sigma_0)) \\
&\leq k^2\delta(\Psi_+, T_G, {T_G}^{n-2}(\sigma_0)) \\
&. \\
&. \\
&. \\
&\leq k^n\delta(\Psi_+, T_G, \sigma_0).
\end{aligned} \tag{6}$$

Also, we know that

$$\begin{aligned}\delta(\Psi_+, T_G, \sigma_0) &= \sup\{\Psi_+(T_G^i(\sigma_0), T_G^j(\sigma_0)) : i, j \in \mathbb{N}\} \\ &= \sup\{\Psi(G^i(\omega_0, \nu_0), G^j(\omega_0, \nu_0)) + \Psi(G^i(\nu_0, \omega_0), G^j(\nu_0, \omega_0))\} \\ &= \delta_G(\Psi, (\omega_0, \nu_0)) + \delta_G(\Psi, (\nu_0, \omega_0)).\end{aligned}$$

Since $\delta_G(\Psi, (\omega_0, \nu_0)) < \infty$ and $\delta_G(\Psi, (\nu_0, \omega_0)) < \infty$, then we have

$$\delta(\Psi_+, T_G, \sigma_0)) < \infty.$$

Using this in (6) , for all $m \in \mathbb{N}$, we obtain

$$\begin{aligned}\Psi_+(\sigma_n, \sigma_{n+m}) &= \Psi_+({T_G}^n(\sigma_0), {T_G}^{n+m}(\sigma_0)) \\ &\leq \delta(\Psi_+, T_G, {T_G}^n(\sigma_0)) \\ &\leq k^n \delta(\Psi_+, T_G, \sigma_0) \to 0 \text{ as } n \to \infty.\end{aligned}$$

Thus, $\{\sigma_n\}$ is a Cauchy sequence. As (Γ^2, Ψ_+) is complete, the sequence $\{\sigma_n\}$ converges to $\tilde{\sigma}$ for some $\tilde{\sigma} = (\tilde{\omega}, \tilde{\nu}) \in \Gamma^2$.

Finally, we need to prove that $\tilde{\sigma} = (\tilde{\omega}, \tilde{\nu})$ is a fixed point of T_G and that it is a CFP of G.

Now, we have two cases to consider about the Cauchy sequence $\{\sigma_n\} = \{(\omega_n, \nu_n)\} \subseteq \Gamma^2$.

Case (1): If $\{\sigma_n\}$ is finite, then there exists $n_0 \in \mathbb{N}$ such that $\sigma_n = T_G^n(\omega_0, \nu_0) = \tilde{\sigma}$ for all $n \geq n_0$. Now,

$$T_G(\tilde{\sigma}) = T_G(\sigma_n) = T_G(T_G^n(\omega_0, \nu_0)) = T_G^{n+1}(\omega_0, \nu_0) = \tilde{\sigma}, \text{ since } n+1 \geq n \geq n_0.$$

Thus, $\tilde{\sigma}$ is a fixed point of T_G.

Case (2): If $\{\sigma_n\} = \{(\omega_n, \nu_n)\}$ is an infinite Cauchy sequence and we suppose that $T_G(\tilde{\sigma}) \neq \tilde{\sigma}$, then since the space (Γ^2, Ψ_+) is subordinate with the function ξ', we have

$$\begin{aligned}\Psi_+(\tilde{\sigma}, T_G(\tilde{\sigma})) &\leq \xi'\left(\limsup_{n\to\infty} \Psi_+(T_G^n(\sigma_0), T_G(\tilde{\sigma}))\right) \\ &\leq \xi'\left(\limsup_{n\to\infty} k\Psi_+(\tilde{\sigma})\right) = \xi'(0) = 0.\end{aligned}$$

This implies that $\tilde{\sigma} = T_G(\tilde{\sigma})$; that is, $\tilde{\sigma}$ is a fixed point of T_G.

By using Remark 3, we can deduce that $\tilde{\sigma} = (\tilde{\omega}, \tilde{\nu})$ is a CFP of G; that is, $\tilde{\omega} = G(\tilde{\omega}, \tilde{\nu})$ and $\tilde{\nu} = G(\tilde{\nu}, \tilde{\omega})$. □

Following this, we state several further requirements for a CFP of G to be a unique.

Theorem 3. *Assume $\tilde{\sigma} = (\tilde{\omega}, \tilde{\nu})$ and $\rho = (\lambda, \mu)$ are CFPs of G such that they are comparable and let $\Psi_+(\rho, \tilde{\sigma}) < \infty$. Then, $\rho = \tilde{\sigma}$.*

Proof. Now,

$$\begin{aligned}\Psi_+(\rho, \tilde{\sigma}) &= \Psi_+(T_G(\rho), T_G(\tilde{\sigma})) \leq k\Psi_+(\rho, \tilde{\sigma}) \\ &\Rightarrow \Psi_+(\rho, \tilde{\sigma}) = 0 \\ &\Rightarrow \rho = \tilde{\sigma} \\ &\Rightarrow (\lambda, \mu) = (\tilde{\omega}, \tilde{\nu}).\end{aligned}$$

As a result, the proof follows. □

Theorem 4. *Assume $\rho = (\lambda, \mu)$ and $\tilde{\sigma} = (\tilde{\omega}, \tilde{\nu})$ are CFPs of G such that they are incomparable. Assume that there is a lower bound or upper bound $\sigma^* = (\omega^*, \nu^*) \in \Gamma^2$ of ρ and $\tilde{\sigma}$ such that $\Psi_+(\rho, \sigma^*) < \infty$ and $\Psi_+(\tilde{\sigma}, \sigma^*) < \infty$. Then, $\rho = \tilde{\sigma}$.*

Proof. It is clear that, for every $n \in \mathbb{N}$, $T^n{}_G(\sigma^*)$ is comparable to $\rho = T^n{}_G(\rho)$ as well as to $\tilde{\sigma} = T^n{}_G(\tilde{\sigma})$. By using the contraction principle (5), we obtain

$$\Psi_+(T_G(\rho), T_G(\sigma^*)) \leq k\Psi_+(\rho, \sigma^*),$$

and

$$\Psi_+(T_G^2(\rho), T_G^2(\sigma^*)) \leq k\Psi_+((T_G(\rho), T_G(\sigma^*)) \leq k^2\Psi_+(\rho, \sigma^*),$$

In a similar way, we obtain

$$\Psi_+(T_G^n(\rho), T_G^n(\sigma^*)) \leq k^n\Psi_+(\rho, \sigma^*). \tag{7}$$

Employing the axioms of Ψ_+-subordinate semimetric spaces and the above inequality, we obtain

$$\Psi_+(\rho, T_G^n(\sigma^*)) \leq \xi'(\limsup_{n\to\infty} \Psi_+(T_G^n(\rho), T_G^n(\sigma^*))) \leq \xi'(\limsup_{n\to\infty} k^n\Psi_+(\rho, \sigma^*)).$$

Since $\Psi_+((\rho, \sigma^*) < \infty$ and $0 < k < 1$, we must have $\Psi_+(\rho, T_G^n(\sigma^*)) \to \xi'(0) = 0$ whenever $n \to \infty$. Thus, the sequence $\{T_G^n(\sigma^*)\}$ also converges to ρ.

Similarly, it can be demonstrated that the sequence $\{T_G^n(\sigma^*)\}$ also converges to $\tilde{\sigma}$. Through Remark 2, we can conclude that $\tilde{\sigma} = \rho$; that is, $(\tilde{\omega}, \tilde{\nu}) = (\lambda, \mu)$. □

Next, we look for further requirements for the equality of CFP components. To demonstrate equality, assume the following conditions:

(Q_1) Assume that $(\tilde{\omega}, \tilde{\nu})$ is a CFP of G with comparable components $\tilde{\omega}$ and $\tilde{\nu}$ in Γ such that $\Psi(\tilde{\omega}, \tilde{\nu}) < \infty$.

(Q_2) Let every pair of components $\omega, \nu \in \Gamma$ have either a lower bound or an upper bound $\rho \in \Gamma$ such that $\Psi(\omega, \rho) < \infty, \Psi(\nu, \rho) < \infty, \Psi(\omega, \nu) < \infty$, and $\Psi(\rho, \rho) < \infty$.

(Q_3) Let ω_0, ν_0 be comparable in Γ with $\Psi(\omega_0, \nu_0) < \infty$.

Theorem 5. *If we add any of the preceding requirements to the hypothesis of Theorem 2, then the components of a CFP are equal.*

Proof. The theorem is proved by the following cases.

Case I: Assume that requirement (Q1) is satisfied, together with the assumptions of Theorem 2. Let $\Sigma = (\tilde{\omega}, \tilde{\nu})$ and $\Delta = (\tilde{\nu}, \tilde{\omega})$. By using the contraction principal in Theorem 2, we obtain

$$\begin{aligned}
&\Rightarrow \Psi(G(\tilde{\omega}, \tilde{\nu}), G(\tilde{\nu}, \tilde{\omega})) + \Psi(G(\tilde{\nu}, \tilde{\omega}), G(\tilde{\omega}, \tilde{\nu})) \leq k(\Psi(\tilde{\omega}, \tilde{\nu}) + \Psi(\tilde{\nu}, \tilde{\omega}))\\
&\Rightarrow \Psi(G(\tilde{\omega}, \tilde{\nu}), G(\tilde{\nu}, \tilde{\omega})) \leq k\Psi(\tilde{\omega}, \tilde{\nu})\\
&\Rightarrow \Psi(\tilde{\omega}, \tilde{\nu}) \leq k\Psi(\tilde{\omega}, \tilde{\nu})\\
&\Rightarrow \Psi(\tilde{\omega}, \tilde{\nu}) = 0, i.e., \tilde{\omega} = \tilde{\nu}.
\end{aligned}$$

Case II: Assume that requirement (Q2) is satisfied, together with the assumptions of Theorem 2. We consider $(\tilde{\omega}, \tilde{\nu})$ to be a CFP of G with $\tilde{\omega}, \tilde{\nu}$ being incomparable.

Suppose $\tilde{\rho} \in \Gamma$ is an upper bound of $\tilde{\omega}$ and $\tilde{\nu}$ such that $\Psi(\tilde{\omega}, \tilde{\rho}) < \infty, \Psi(\tilde{\nu}, \tilde{\rho}) < \infty$, $\Psi(\tilde{\omega}, \tilde{\nu}) < \infty$, and $\Psi(\tilde{\rho}, \tilde{\rho}) < \infty$.

Then, $\tilde{\omega} \leq \tilde{\rho}$ and $\tilde{\nu} \leq \tilde{\rho}$. With respect to partial order in (Γ^2, Ψ_+), we obtain

$$(\tilde{\omega}, \tilde{\nu}) \geq (\tilde{\omega}, \tilde{\rho}); (\tilde{\omega}, \tilde{\rho}) \leq (\tilde{\rho}, \tilde{\omega}); (\tilde{\rho}, \tilde{\omega}) \geq (\tilde{\nu}, \tilde{\omega}).$$

Let $\Sigma = (\tilde{\omega}, \tilde{\nu})$ and $\Delta = (\tilde{\omega}, \tilde{\rho})$. Because Σ and Δ are comparable due to the conditions of contraction (3) and (5), we have

$$\begin{aligned}
&\Psi(G(\tilde{\omega}, \tilde{\nu}), G(\tilde{\omega}, \tilde{\rho})) + \Psi(G(\tilde{\nu}, \tilde{\omega}), G(\tilde{\rho}, \tilde{\omega})) \leq k[\Psi(\tilde{\omega}, \tilde{\omega}) + \Psi(\tilde{\nu}, \tilde{\rho})] \\
&\Rightarrow \Psi_+(T_G(\Sigma), T_G(\Delta)) \leq k[\Psi(\tilde{\omega}, \tilde{\omega}) + \Psi(\tilde{\nu}, \tilde{\rho})].
\end{aligned}$$

Using Proposition 2, we must have $\Psi_+(\tilde{\omega}, \tilde{\omega}) = 0$, and thus, we obtain

$$\Psi_+(T_G(\Sigma), T_G(\Delta)) \leq k\Psi(\tilde{\nu}, \tilde{\rho}). \tag{8}$$

Now, as $\Sigma = (\tilde{\omega}, \tilde{\nu})$ is a fixed point of operator T_G, $T_G^n(\Sigma) = \Sigma$ for each $n \in \mathbb{N}$. Then, inequality (8) is simplified to

$$\begin{aligned}
&\Psi_+(T_G^n(\Sigma), T_G^n(\Delta)) \leq k^n \Psi_+(\Sigma, \Delta) \\
&\Rightarrow \Psi_+(\Sigma, T_G^n(\Delta)) \leq k^n \Psi(\tilde{\nu}, \tilde{\rho}) \\
&\Rightarrow \Psi_+(\Sigma, T_G^n(\Delta)) \to 0
\end{aligned} \tag{9}$$

as $n \to \infty$ and $\Psi(\tilde{\nu}, \tilde{\rho}) < \infty$. Hence, the sequence $(T_G^n(\Delta))$ converges to Σ.

Next, let $\Xi = (\tilde{\nu}, \tilde{\omega})$ and $\Pi = (\tilde{\rho}, \tilde{\omega})$. Clearly Ξ and Π are comparable, and thus, we obtain

$$\begin{aligned}
\Psi_+(T_G(\Xi), T_G(\Pi)) &\leq k\Psi_+(\Xi, \Pi) \\
&\leq k[\Psi_+((\tilde{\nu}, \tilde{\omega}), (\tilde{\rho}, \tilde{\omega}))] \\
&\leq k[\Psi(\tilde{\nu}, \tilde{\omega}) + \Psi(\tilde{\omega}, \tilde{\omega})] \\
&\leq k\Psi(\tilde{\nu}, \tilde{\rho}).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\Psi_+(T_G^n(\Xi), T_G^n(\Pi)) \leq k^n \Psi_+(\Xi, \Pi) \\
&\qquad \Rightarrow \Psi_+(\Xi, T_G^n(\Pi)) \leq k^n \Psi(\tilde{\nu}, \tilde{\rho}) \Rightarrow \Psi_+(\Xi, T_G^n(\Pi)) \to 0
\end{aligned} \tag{10}$$

as $n \to \infty$ and $\Psi(\tilde{\nu}, \tilde{\rho}) < \infty$. Thus, the sequence $(T_G^n(\Pi))$ is convergent to Ξ.

Now, as Δ and Π are comparable, then by the condition of contraction (3), we obtain

$$\begin{aligned}
&\Psi_+(T_G^n(\Delta), T_G^n(\Pi)) \leq k^n \Psi_+(\Delta, \Pi) \\
&\Rightarrow \Psi_+(T_G^n(\Delta), T_G^n(\Pi)) \leq k^n (\Psi_+((\tilde{\omega}, \tilde{\rho}), (\tilde{\rho}, \tilde{\omega}))) \\
&\Rightarrow \Psi_+(T_G^n(\Delta), T_G^n(\Pi)) \leq k^n \{\Psi(\tilde{\omega}, \tilde{\rho}) + \Psi(\tilde{\rho}, \tilde{\omega})\} \\
&\Rightarrow \Psi_+(T_G^n(\Delta), T_G^n(\Pi)) \to 0
\end{aligned} \tag{11}$$

as $n \to \infty$ since $\Psi(\tilde{\omega}, \tilde{\rho}) < \infty$ and $\Psi(\tilde{\rho}, \tilde{\omega}) < \infty$. In using the axioms of Ψ_+-subordinate semimetric spaces along with (9), (10), and (11), there exists a non-decreasing function ξ' with $\lim\limits_{\omega \to 0} \xi'(\omega) = 0$ such that

$$\begin{aligned}
\Psi_+(\Sigma, \Xi) &\leq \xi'(\limsup_{n \to \infty} \Psi_+(T_G^n(\Delta), T_G^n(\Pi))) = \xi'(0) = 0 \\
&\Rightarrow \Sigma = \Xi \\
&\Rightarrow (\tilde{\omega}, \tilde{\nu}) = (\tilde{\nu}, \tilde{\omega}) \\
&\Rightarrow \tilde{\omega} = \tilde{\nu}.
\end{aligned}$$

Alternatively, it is easy to show that the components of a fixed point are equal by assuming $\tilde{\rho} \in \Gamma$ is a lower bound of ω and ν such that $\Psi(\tilde{\omega}, \tilde{\nu}) < \infty, \Psi(\tilde{\nu}, \tilde{\rho}) < \infty, \Psi(\tilde{\omega}, \tilde{\nu}) < \infty$, and $\Psi(\tilde{\rho}, \tilde{\rho}) < \infty$.

Case III: Assume that requirement (Q3) is satisfied, together with the assumptions of Theorem 2. Since G is an MM operator, for each $n \geq 1$, $\omega_n = G(\omega_{n-1}, \nu_{n-1})$ and $\nu_n = G(\nu_{n-1}, \omega_{n-1})$ are comparable and $\omega_n \to \tilde{\omega}$, and $\nu_n \to \tilde{\nu}$ as $n \to \infty$. Using the axioms of subordinate semimetric spaces, we obtain

$$\Psi(\tilde{\omega}, \tilde{\nu}) \leq \xi(\limsup_{n\to\infty} \Psi(\omega_n, \nu_n)). \tag{12}$$

Again, let $\Sigma = \{\omega_n, \nu_n\}$ and $\Delta = \{\nu_n, \omega_n\}$ in the condition of contraction of Theorem 2; then, for each $n \geq 0$, we have

$$\begin{aligned} &\Psi(G(\omega_n, \nu_n), G(\nu_n, \omega_n)) \leq k\Psi(\omega_n, \nu_n) \\ &\quad \Rightarrow \Psi(\omega_{n+1}, \nu_{n+1}) \leq k\Psi(\omega_n, \nu_n). \end{aligned} \tag{13}$$

By inequalities (12) and (13), we obtain

$$\Psi(\tilde{\omega}, \tilde{\nu}) \leq \xi(\limsup_{n\to\infty} \Psi(\omega_n, \nu_n)) \leq \xi(\limsup_{n\to\infty} k^n \Psi(\omega_0, \nu_0)) = \xi(0) = 0$$

as $n \to \infty$. This implies that $\Psi(\tilde{\omega}, \tilde{\nu}) = 0$. As a result, we must have $\tilde{\omega} = \tilde{\nu}$. □

The following corollary is a new form of Theorem (2.1.6) in [6].

Corollary 1. *Let Γ be a PO complete subordinate semimetric space. Assume that the mapping $G : \Gamma^2 \to \Gamma$ satisfies the MM property on Γ, and there is a $k \in (0,1)$ with*

$$\Psi((G(\omega, \nu), G(\lambda, \mu)) \leq \frac{k}{2}\Psi_+((\omega, \nu), (\lambda, \mu))$$

for all $\omega \geq \lambda$ and $\nu \leq \mu$. Also, consider that there exists $\omega_0, \nu_0 \in \Gamma$ such that the following hold:

(i) $\omega_0 \leq G(\omega_0, \nu_0)$ *and* $\nu_0 \geq G(\nu_0, \omega_0)$*;*

(ii) $\delta_G(\Psi, (\omega_0, \nu_0)) < \infty$ *and* $\delta_G(\Psi, (\nu_0, \omega_0)) < \infty$*.*

Then, there exists $\omega, \nu \in \Gamma$ such that $\omega = G(\omega, \nu); \nu = G(\nu, \omega)$.

Remark 4. *To prove the presence of CFPs, the authors of [4] investigated two different assumptions. The first assumption is that the function G is continuous and the second assumption is if $\{\omega_n\}$ and $\{\nu_n\}$ are non-increasing and non-decreasing sequences, respectively, such that $\{\omega_n\} \to \omega$ and $\{\nu_n\} \to \nu$, it follows that $\omega_n \leq \omega$ and $\nu_n \leq \nu$ for all $n \in \mathbb{N}$. However, Corollary 1 guarantees the presence of CFPs without requiring any of the preceding assumptions.*

Remark 5. *Since each b-metric space is a subordinate semimetric space such that $\xi(\omega) = s(\omega) \geq 1$ in Definition 4, it is easy to prove the CFP results in a PO b-metric space based on this paper's findings. In particular, the CFP findings in a b-metric space can be deduced from Theorem (2.2) in [7] using Corollary 1.*

Remark 6. *In Corollary 1, the quality of the components of a CFP and the uniqueness of a CFP of G are ensured using Theorems 3–5 as well.*

Similarly, anyone can also prove the presence of a CFP of Ψ_m on Γ^2. The next theorem presented addresses this.

Theorem 6. *Assume that the mapping $G : \Gamma^2 \to \Gamma$ satisfies the MM property on Γ and there is a $k \in (0,1)$ with*

$$\Psi_m((G(\omega, \nu), G(\nu, \omega)), (G(\lambda, \mu), G(\mu, \lambda))) \leq k\Psi_m((\omega, \nu), (\lambda, \mu))$$

for all $\omega \geq \lambda$ and $\nu \leq \mu$. If there exist $\omega_0, \nu_0 \in \Gamma$ such that the following hold:

(i) $\omega_0 \leq G(\omega_0, \nu_0)$ *and* $\nu_0 \geq G(\nu_0, \omega_0)$*;*

(ii) $\delta_G(\Psi,(\omega_0,\nu_0)) < \infty$ *and* $\delta_G(\Psi,(\nu_0,\omega_0)) < \infty$;

then G *has a CFP* $(\omega,\nu) \in \Gamma^2$*; that is,* $\omega = G(\omega,\nu)$ *and* $\nu = G(\nu,\omega)$*.*

Proof. The proof is essentially the same as the proof of Theorem 2. Hence, we will skip the proof. □

We will now present examples to support our major conclusion.

Example 1. *Let* $\Gamma = [0,1]$*. Let* $\Psi : \Gamma \times \Gamma \to [0,\infty]$ *be given by* $\Psi(\omega,\nu) = (\omega-\nu)^2$*.*
Then, (Γ,Ψ) *is a subordinate semimetric space to* $\xi(t) = t$, $t \in [0,\infty]$*.*
Consider the subordinate semimetric space on (Γ^2,Ψ_+)*, where*

$$\Psi_+((\omega,\nu),(\mu,\lambda)) = \Psi(\omega,\mu) + \Psi(\nu,\lambda).$$

Define $G : \Gamma^2 \to \Gamma$ *as* $G(\omega,\nu) = \begin{cases} 0, & \text{if } 2\omega \le \nu; \\ \frac{2\omega-\nu}{5}, & \text{otherewise.} \end{cases}$

1. G ***has the MM property****.*

   (MM-1) Let $\omega_1 \le \omega_2$*. For all* $\nu \in \Gamma$*, consider the following.*
   Since $2\omega_1 - \nu \le 2\omega_2 - \nu$*, then* $G(\omega_1,\nu) \le G(\omega_2,\nu)$*.*
   Thus, G *is monotonically non-decreasing in its first component.*
   (MM-2) Let $\nu_1 \le \nu_2$*. For all* $\omega \in \Gamma$*, if* $\nu_2 < 2\omega$*, consider the following.*
   Since $2\omega - \nu_1 \ge 2\omega - \nu_2$*, then* $G(\omega,\nu_1) \ge G(\omega,\nu_1)$*.*
   If $\nu_2 \ge 2\omega$*, then* $G(\omega,\nu_1) = G(\omega,\nu_1) = 0$*.*
   Thus, for all $\omega \in \Gamma$*,* $G(\omega,\nu_1) \ge G(\omega,\nu_1)$*.*
   Thus, G *is monotonically non-increasing in its second component.*

2. $\delta_G(\Psi,(\omega_0,\nu_0)) < \infty$ ***and*** $\delta_G(\Psi,(\nu_0,\omega_0)) < \infty$*.*
   Let $(\omega_0,\nu_0) = (1,0)$*. Then,*

   $$\omega_1 = G(\omega_0,\nu_0) = G(1,0) = \frac{2}{5} \le 1 = \omega_0 \text{ and } \nu_1 = G(\nu_0,\omega_0) = G(0,1) = 0 \ge 0 = \nu_0.$$

   $$\text{Also, } \omega_2 = G^2(\nu_0,\omega_0) = G(\omega_1,\nu_1) = G(\frac{2}{5},0) = \frac{4}{25} = (\frac{2}{5})^2 \le \frac{2}{5} = \omega_1 \text{ and}$$

   $$\nu_2 = G^2(\nu_0,\omega_0) = G(\nu_1,\omega_1) = G(0,\frac{2}{5}) = 0 \ge 0 = \nu_1.$$

   $$\text{Thus, } G^n(\omega_0,\nu_0) = \left(\frac{2}{5}\right)^n, \text{ and } G^n(\nu_0,\omega_0) = 0.$$

   Thus, $\delta_G(\Psi,(\omega_0,\nu_0)) < \infty$ *and* $\delta_G(\Psi,(\nu_0,\omega_0)) < \infty$*.*

3. G ***satisfies the contraction condition****.*
   Let $(\omega,\nu),(\mu,\lambda) \in \Gamma^2$ *with* $\omega \ge \mu$ *and* $\nu \le \lambda$*.*
   (a) *Suppose* $\omega = \mu = 0$*. Then, for all* $\nu,\lambda \in \Gamma$*, we have* $\Psi(G(\omega,\nu),G(\mu,\lambda)) = 0$*. Hence,*

   $$\begin{aligned} \Psi(G(\omega,\nu),G(\mu,\lambda)) + \Psi(G(\nu,\omega),G(\lambda,\mu)) &= \Psi(G(\nu,\omega),G(\lambda,\mu)) \\ &= \Psi(\frac{2\nu}{5},\frac{2\lambda}{5}) \\ &= \left(\frac{2\nu}{5} - \frac{2\lambda}{5}\right)^2 \\ &= \frac{4}{25}\left[(\nu-\lambda)^2 + (\omega-\mu)^2\right] \\ &\le \frac{4}{25}[\Psi(\omega,\mu) + \Psi(\nu,\lambda)] \\ &\le \frac{4}{25}\Psi_+((\omega,\nu),(\mu,\lambda)). \end{aligned}$$

(b) *For* $\nu = \lambda = 0$, *then, similarly,* $\Psi(G(\omega,\nu), G(\mu,\lambda)) + \Psi(G(\nu,\omega), G(\lambda,\mu)) \leq \frac{4}{25}\Psi_+((\omega,\nu),(\mu,\lambda))$.

(c) *For* $\mu = 0$, $\omega \neq 0$, *and* $\nu = 0$, $\lambda \neq 0$, *in a similar way, we have*

$$\Psi(G(\omega,\nu), G(\mu,\lambda)) + \Psi(G(\nu,\omega), G(\lambda,\mu)) \leq \frac{4}{25}\Psi_+((\omega,\nu),(\mu,\lambda)).$$

(d) *For* $(\omega,\nu),(\mu,\lambda) \in \Gamma^2$ *with* $\omega \geq \mu$ *and* $\nu \leq \lambda$ *and* $\mu \neq 0$ *and* $\nu \neq 0$, *note that* $(a-b)^2 \leq 2(a^2+b^2)$, $a,b \in \mathbb{R}$. *We then have*

$$\begin{aligned}
\Psi(G(\omega,\nu), G(\mu,\lambda)) + \Psi(G(\nu,\omega), G(\lambda,\mu)) &= \Psi\left(\frac{2\omega-\nu}{5}, \frac{2\mu-\lambda}{5}\right) + \Psi\left(\frac{2\nu-\omega}{5}, \frac{2\lambda-\mu}{5}\right) \\
&= \left(\frac{2\omega-\nu}{5} - \frac{2\mu-\lambda}{5}\right)^2 + \left(\frac{2\nu-\omega}{5} - \frac{2\lambda-\mu}{5}\right)^2 \\
&= \left(\frac{2(\omega-\mu)}{5} - \frac{(\nu-\lambda)}{5}\right)^2 + \left(\frac{2(\nu-\lambda)}{5} - \frac{(\omega-\mu)}{5}\right)^2 \\
&\leq 2\left[\frac{4(\omega-\mu)^2}{25} + \frac{(\nu-\lambda)^2}{25}\right] + 2\left[\frac{4(\nu-\lambda)^2}{25} + \frac{(\omega-\mu)^2}{25}\right] \\
&= 2\left[\frac{(\omega-\mu)^2}{5} + \frac{(\nu-\lambda)^2}{5}\right] \\
&= \frac{2}{5}\left[(\omega-\mu)^2 + (\nu-\lambda)^2\right] \\
&\leq \frac{4}{25}[\Psi(\omega,\mu) + \Psi(\nu,\lambda)] \\
&\leq \frac{4}{25}\Psi_+((\omega,\nu),(\mu,\lambda)).
\end{aligned}$$

Thus, G satisfies the contraction condition.
Hence, the point $(0,0)$ *is the only coupled fixed point of G.*

Example 2. *Let* $\Gamma = [0,1]$. *Let* $\Psi : \Gamma \times \Gamma \to [0,\infty]$ *be given by*

$$\Psi(\omega,\nu) = \Psi(\nu,\omega) = \begin{cases} n^2, & \text{if } (\omega,\nu) = \left(\frac{1}{n}, 0\right), n \in \mathbb{N}; \\ n, & \text{if } (\omega,\nu) = \left(\frac{r}{n(r+1)}, 0\right), n,r \in \mathbb{N}; \\ (x-y)^2, & \text{otherewise.} \end{cases}$$

Let $m \in \mathbb{N}$.
Note that $\lim_{n\to\infty} \Psi\left(\frac{1}{m}, \frac{n}{m(n+1)}\right) = \lim_{n\to\infty}\left(\frac{1}{m} - \frac{n}{m(n+1)}\right)^2 = 0$, *and*

$$\lim_{n,r\to\infty} \Psi\left(\frac{n}{m(n+1)}, \frac{r}{m(r+1)}\right) = \lim_{n,r\to\infty}\left(\frac{n}{m(n+1)} - \frac{r}{m(r+1)}\right)^2 = 0.$$

Thus, the sequence $\left\{\frac{n}{m(n+1)}\right\}_{n\in\mathbb{N}}$ *is an infinite Cauchy sequence that is convergent to* $\frac{1}{m}$. *Now, suppose there is a* $c > 0$ *such that*

$$m^2 = \Psi\left(\frac{1}{m}, 0\right) \leq c \limsup_{n\to\infty} \Psi\left(\frac{n}{m(n+1)}, 0\right) = cm,$$

then $c \geq m$ *for all* $m \in \mathbb{N}$. *Hence,* (Γ,Ψ) *is not an RS-space. Note that* (Γ,Ψ) *is subordinately semimetric to* $\xi(t) = \begin{cases} t, & 0 \leq t \leq 1; \\ t^4, & t > 1. \end{cases}$

Let s be a real number such that $s > 1$.

Define $G : \Gamma^2 \to \Gamma$ *by* $G(\omega,\nu) = \begin{cases} 0, & \text{if } \omega \leq \nu, \omega = \frac{1}{n}, \nu = 0, n \in \mathbb{N} \\ & \text{or } \omega = \frac{r}{n(r+1)}, \nu = 0, n,r \in \mathbb{N}; \\ \frac{\omega-\nu}{s}, & \text{otherewise.} \end{cases}$

1. *G* ***has MM property***.

   (MM-1) *Let* $\omega_1 \leq \omega_2$. *For all* $\nu \in \Gamma$, *consider the following.*
   Since $\frac{\omega_1-\nu}{s} \leq \frac{\omega_2-\nu}{s}$, *then* $G(\omega_1, \nu) \leq G(\omega_2, \nu)$.
   Thus, G is monotonically non-decreasing in its first component.

   (MM-2) *Let* $\nu_1 \leq \nu_2$. *For all* $\omega \in \Gamma$, *if* $\nu_2 < \omega$, *consider the following.*
   Since $\frac{\omega-\nu_1}{s} \geq \frac{\omega-\nu_2}{s}$, *then* $G(\omega, \nu_1) \geq G(\omega, \nu_1)$.
   If $\nu_2 \geq \omega$, *then* $G(\omega, \nu_1) = G(\omega, \nu_1) = 0$.
   Thus, for all $\omega \in \Gamma$, $G(\omega, \nu_1) \geq G(\omega, \nu_1)$.
   Thus, G is monotonically non-increasing in its second component.

2. $\delta_G(\Psi, (\omega_0, \nu_0)) < \infty$ ***and*** $\delta_G(\Psi, (\nu_0, \omega_0)) < \infty$.
   Let $(\omega_0, \nu_0) = (1, 0)$.

   $$\omega_1 = G(\omega_0, \nu_0) = G(1, 0) = \frac{1}{s} \leq 1 = \omega_0 \text{ and } \nu_1 = G(\nu_0, \omega_0) = G(0, 1) = 0 \geq 0 = \nu_0.$$

   $$\text{Also, } \omega_2 = G^2(\nu_0, \omega_0) = G(\omega_1, \nu_1) = G(\frac{1}{s}, 0) = \frac{1}{s^2} = (\frac{1}{s})^2 \leq \frac{1}{s} = \omega_1 \text{ and}$$
   $$\nu_2 = G^2(\nu_0, \omega_0) = G(\nu_1, \omega_1) = G(0, \frac{1}{s}) = 0 \geq 0 = \nu_1.$$

   Thus, $G^n(\omega_0, \nu_0) = \left(\frac{1}{s}\right)^n$ *and* $G^n(\nu_0, \omega_0) = 0$.
   Thus, $\delta_G(\Psi, (\omega_0, \nu_0)) < \infty$ *and* $\delta_G(\Psi, (\nu_0, \omega_0)) < \infty$.

3. *G* ***satisfies the contraction condition***.
   Let $(\omega, \nu), (\mu, \lambda) \in \Gamma^2$ *with* $\omega \geq \mu$ *and* $\nu \leq \lambda$.

   (a) *Suppose* $\omega = \mu = 0$. *Then, for all* $\nu, \lambda \in \Gamma$, *we have* $\Psi(G(\omega, \nu), G(\mu, \lambda)) = 0$. *Hence,*

   $$\begin{aligned} \Psi(G(\omega,\nu),G(\mu,\lambda)) + \Psi(G(\nu,\omega),G(\lambda,\mu)) &= \Psi(G(\nu,\omega),G(\lambda,\mu)) \\ &= \Psi(\frac{\nu}{s},\frac{\lambda}{s}) \\ &= \left(\frac{\nu}{s} - \frac{\lambda}{s}\right)^2 \\ &= \frac{1}{s^2}\left[(\nu-\lambda)^2 + (\omega-\mu)^2\right] \\ &\leq \frac{1}{s^2}[\Psi(\omega,\mu) + \Psi(\nu,\lambda)] \\ &\leq \frac{1}{s^2}\Psi_+((\omega,\nu),(\mu,\lambda)). \end{aligned}$$

   (b) *For* $\nu = \lambda = 0$, *similarly,* $\Psi(G(\omega, \nu), G(\mu, \lambda)) + \Psi(G(\nu, \omega), G(\lambda, \mu)) \leq \frac{1}{s^2}\Psi_+ ((\omega, \nu), (\mu, \lambda))$.

   (c) *For* $\mu = 0$, $\omega \neq 0$ *and* $\nu = 0$, $\lambda \neq 0$, *in a similar way, we have*
   $\Psi(G(\omega, \nu), G(\mu, \lambda)) + \Psi(G(\nu, \omega), G(\lambda, \mu)) \leq \frac{1}{s^2}\Psi_+((\omega, \nu), (\mu, \lambda))$.

   (d) *For* $(\omega, \nu), (\mu, \lambda) \in \Gamma^2$ *with* $\omega \geq \mu$ *and* $\nu \leq \lambda$ *and* $\mu \neq 0$ *and* $\nu \neq 0$, *note that* $(a - b)^2 \leq 2(a^2 + b^2)$, $a, b \in \mathbb{R}$. *We then have*

$$\begin{aligned}
\Psi(G(\omega,\nu),G(\mu,\lambda))+\Psi(G(\nu,\omega),G(\lambda,\mu)) &= \Psi\left(\frac{\omega-\nu}{s},\frac{\mu-\lambda}{s}\right)+\Psi\left(\frac{\nu-\omega}{s},\frac{\lambda-\mu}{s}\right)\\
&= \left(\frac{\omega-\nu}{s}-\frac{\mu-\lambda}{s}\right)^2+\left(\frac{\nu-\omega}{s}-\frac{\lambda-\mu}{s}\right)^2\\
&= \left(\frac{(\omega-\mu)}{s}-\frac{(\nu-\lambda)}{s}\right)^2+\left(\frac{(\nu-\lambda)}{s}-\frac{(\omega-\mu)}{s}\right)^2\\
&\le 2\left[\frac{(\omega-\mu)^2}{s^2}+\frac{(\nu-\lambda)^2}{s^2}\right]+2\left[\frac{(\nu-\lambda)^2}{s^2}+\frac{(\omega-\mu)^2}{s^2}\right]\\
&= 2\left[\frac{2(\omega-\mu)^2}{s^2}+\frac{2(\nu-\lambda)^2}{s^2}\right]\\
&= \frac{4}{s^2}\left[(\omega-\mu)^2+(\nu-\lambda)^2\right]\\
&\le \frac{4}{s^2}[\Psi(\omega,\mu)+\Psi(\nu,\lambda)]\\
&\le \frac{4}{s^2}\Psi_+((\omega,\nu),(\mu,\lambda)).
\end{aligned}$$

Thus, G satisfies the contraction condition.
Hence, the point $(0,0)$ is the only coupled fixed point of G.

Example 3. *Let $\Gamma=\mathbb{R}\cup\{\infty,-\infty\}$. Let $\Psi:\Gamma\times\Gamma\to[0,\infty]$ be given by*

$$\Psi(\omega,\nu)=\Psi(\nu,\omega)=\begin{cases} n^2, & \text{if } (\omega,\nu)=\left(\frac{1}{n},0\right), n\in\mathbb{N};\\ n, & \text{if } (\omega,\nu)=\left(\frac{r}{n(r+1)},0\right), n,r\in\mathbb{N};\\ 0, & \text{if } (\omega,\nu)=(\infty,\infty), \text{ or } (\omega,\nu)=(-\infty,-\infty)\ ;\\ (x-y)^2, & \text{otherewise.}\end{cases}$$

Let $m\in\mathbb{N}$.
Note that $\lim_{n\to\infty}\Psi\left(\frac{1}{m},\frac{n}{m(n+1)}\right)=\lim_{n\to\infty}\left(\frac{1}{m}-\frac{n}{m(n+1)}\right)^2=0$, and

$$\lim_{n,r\to\infty}\Psi\left(\frac{n}{m(n+1)},\frac{r}{m(r+1)}\right)=\lim_{n,r\to\infty}\left(\frac{n}{m(n+1)}-\frac{r}{m(r+1)}\right)^2=0.$$

Thus, the sequence $\left\{\frac{n}{m(n+1)}\right\}_{n\in\mathbb{N}}$ is an infinite Cauchy sequence that is convergent to $\frac{1}{m}$. Now, suppose there is a $c>0$ such that

$$m^2=\Psi\left(\frac{1}{m},0\right)\le c\limsup_{n\to\infty}\Psi\left(\frac{n}{m(n+1)},0\right)=cm,$$

then $c\ge m$ for all $m\in\mathbb{N}$. Hence, (Γ,Ψ) is not an RS-space. Note that (Γ,Ψ) is subordinately semimetric to $\xi(t)=\begin{cases} t, & 0\le t\le 1;\\ t^2, & t>1.\end{cases}$

Let s be an irrational real number such that $s>\frac{3+\sqrt{5}}{2}>\frac{5}{2}$. Note that $s^2-3s+1>0$, if $s>\frac{3+\sqrt{5}}{2}$; hence, $2s-1<s^2-s$.

Define $G:\Gamma^2\to\Gamma$ by $G(\omega,\nu)=\begin{cases} 0, & \text{if } \omega=\nu=\infty \text{ or } \omega=\nu=-\infty\ ;\\ \frac{\omega-\nu}{s}, & \text{otherwise.}\end{cases}$

1. *G **has MM property**.*

   (MM-1) Let $\omega_1\le\omega_2$. For all $\nu\in\Gamma$, consider the following.
   Since $\frac{\omega_1-\nu}{s}\le\frac{\omega_2-\nu}{s}$, then $G(\omega_1,\nu)\le G(\omega_2,\nu)$.

Thus G, is monotonically non-decreasing in its first component.

(MM-2) Let $\nu_1 \leq \nu_2$. For all $\omega \in \Gamma$,

Since $\frac{\omega-\nu_1}{s} \geq \frac{\omega-\nu_2}{s}$, then $G(\omega,\nu_1) \geq G(\omega,\nu_1)$.

Thus, for all $\omega \in \Gamma$, $G(\omega,\nu_1) \geq G(\omega,\nu_1)$.

Thus, G is monotonically non-increasing in its second component.

2. $\delta_G(\Psi,(\omega_0,\nu_0)) < \infty$ ***and*** $\delta_G(\Psi,(\nu_0,\omega_0)) < \infty$.

Let $(\omega_0,\nu_0) = (-s, s-1)$. Then,

$$\omega_1 = G(\omega_0,\nu_0) = G(-s,s-1) = \frac{-2s+1}{s} > \frac{-s^2+s}{s} = -s+1 > -s = \omega_0$$

and

$$\nu_1 = G(\nu_0,\omega_0) = G(s-1,-s) = \frac{2s-1}{s} < \frac{s^2-s}{s} = s-1 = \nu_0.$$

Also,

$$\omega_2 = G^2(\nu_0,\omega_0) = G(\omega_1,\nu_1) = G(\frac{-2s+1}{s}, \frac{2s-1}{s}) = \frac{-4s+2}{s^2} = \frac{2}{s}\left(\frac{-2s+1}{s}\right) = \frac{2}{s}\omega_1 > \frac{-2s+1}{s} = \omega_1$$

and

$$\nu_2 = G^2(\nu_0,\omega_0) = G(\nu_1,\omega_1) = G(\frac{2s-1}{s}, \frac{-2s+1}{s}) = \frac{4s-2}{s^2} = \frac{2}{s}\left(\frac{2s-1}{s}\right) = \frac{2}{s}\nu_1 < \frac{2s-1}{s} = \nu_1.$$

Thus,

$$G^n(\omega_0,\nu_0) = \left(\frac{2}{s}\right)^{n-1}\omega_1, \quad G^n(\nu_0,\omega_0) = \left(\frac{2}{s}\right)^{n-1}\nu_1.$$

Thus, $\delta_G(\Psi,(\omega_0,\nu_0)) < \infty$ and $\delta_G(\Psi,(\nu_0,\omega_0)) < \infty$.

3. *G* ***satisfies the contraction condition****.*

Let $(\omega,\nu),(\mu,\lambda) \in \Gamma^2$ with $\omega \geq \mu$ and $\nu \leq \lambda$.

Note that $(a-b)^2 \leq 2(a^2+b^2)$, $a,b \in \mathbb{R}$. We then have

$$\begin{aligned}
\Psi(G(\omega,\nu),G(\mu,\lambda)) + \Psi(G(\nu,\omega),G(\lambda,\mu)) &= \Psi\left(\frac{\omega-\nu}{s}, \frac{\mu-\lambda}{s}\right) + \Psi\left(\frac{\nu-\omega}{s}, \frac{\lambda-\mu}{s}\right) \\
&= \left(\frac{\omega-\nu}{s} - \frac{\mu-\lambda}{s}\right)^2 + \left(\frac{\nu-\omega}{s} - \frac{\lambda-\mu}{s}\right)^2 \\
&= \left(\frac{(\omega-\mu)}{s} - \frac{(\nu-\lambda)}{s}\right)^2 + \left(\frac{(\nu-\lambda)}{s} - \frac{(\omega-\mu)}{s}\right)^2 \\
&\leq 2\left[\frac{(\omega-\mu)^2}{s^2} + \frac{(\nu-\lambda)^2}{s^2}\right] + 2\left[\frac{(\nu-\lambda)^2}{s^2} + \frac{(\omega-\mu)^2}{s^2}\right] \\
&= 2\left[\frac{2(\omega-\mu)^2}{s^2} + \frac{2(\nu-\lambda)^2}{s^2}\right] \\
&= \frac{4}{s^2}\left[(\omega-\mu)^2 + (\nu-\lambda)^2\right] \\
&\leq \frac{4}{s^2}[\Psi(\omega,\mu) + \Psi(\nu,\lambda)] \\
&\leq \frac{4}{s^2}\Psi_+((\omega,\nu),(\mu,\lambda)).
\end{aligned}$$

Thus, G satisfies the contraction condition.

Hence, the point $(0,0)$ is a coupled fixed point of G. Also, the point $(\infty,-\infty)$ is a coupled fixed point of G as well.

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