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# The General Solution to a Classical Matrix Equation $AXB = C$ over the Dual Split Quaternion Algebra

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**Abstract:** In this paper, we investigate the necessary and sufficient conditions for solving a dual split quaternion matrix equation $AXB = C$, and present the general solution expression when the solvability conditions are met. As an application, we delve into the necessary and sufficient conditions for the existence of a Hermitian solution to this equation by using a newly defined real representation method. Furthermore, we obtain the solutions for the dual split quaternion matrix equations $AX = C$ and $XB = C$. Finally, we provide a numerical example to demonstrate the findings of this paper.

**Keywords:** dual split quaternion; real representation; matrix equation; general solution

**MSC:** 15A03; 15A09; 15A24; 15B33; 15B57

## 1. Introduction

In 1843, Hamilton [1] introduced the real quaternions, which can be represented as

$$\mathbb{H} = \left\{ q = q_0 + q_1 i + q_2 j + q_3 k : i^2 = j^2 = k^2 = ijk = -1, q_0, q_1, q_2, q_3 \in \mathbb{R} \right\}. \tag{1}$$

The set of real quaternions form a noncommutative division algebra [2,3]. In 1849, James Cockle [4] introduced split quaternions:

$$\mathbb{H}_s = \{ q = q_0 + q_1 i + q_2 j + q_3 k : q_0, q_1, q_2, q_3 \in \mathbb{R} \}, \tag{2}$$

where

$$i^2 = -j^2 = -k^2 = -1, ij = -ji = k, jk = -kj = -i, ki = -ik = j. \tag{3}$$

The set of split quaternions comprises a four-dimensional associative and noncommutative Clifford algebra that is characterized by the existence of zero divisors, nilpotent elements, and nontrivial idempotents, as referenced in [5–7]. This algebra has found widespread application in the fields of geometry and physics, as evidenced by works such as [8–10]. In 1873, Clifford introduced the concept of dual numbers, which is an expansion of the real numbers by adjoining a new element $\epsilon$ with the property $\epsilon^2 = 0$ [11]. The set of dual numbers forms a two-dimensional commutative and associative algebra over real numbers. As an extension of quaternions through dual number coefficients, dual quaternions have proven useful in theoretical kinematics, as well as in practical applications, like 3D computer graphics, robotics, and computer vision [12–14]. Similarly, we can extend split quaternions by incorporating dual numbers. This concept has numerous applications in screw motions and curve theory within the three-dimensional Minkowski space, piquing the interest of numerous scholars, as demonstrated in [15–18].

In [17], the components of a dual split quaternion are obtained by replacing the L-Euler parameters with their split dual versions. In [19], Kong et al. gave three forms of De Moivre's theorem for the representation matrix of dual split quaternions by using the

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polar representation of dual split quaternions. In [20], authors use dual split quaternions to represent involution and anti-involution mappings. Some important properties and some interesting results of matrices over dual split quaternions are presented in [21]. Furthermore, Ref. [18] explored the dual split quaternionic representation of general displacement.

It is well established that linear matrix equations have been a focal point in matrix theory and its applications. Numerous researchers have devoted attention to studying the solutions of matrix equations [22–26]. The matrix equation

$$AXB = C \tag{4}$$

is a classical and fundamental topic that has been extensively investigated, yielding a series of significant results. For instance, Ben-Israel and Greville [27] provided the necessary and sufficient conditions for successfully solving matrix Equation (4). Liao et al. [28] studied the centrally symmetric solutions of matrix Equation (4) when $B = A^T$. Huang et al. [29] investigated the skew-symmetric solution and the optimal approximate solution for matrix Equation (4). Peng [30] studied the centro-symmetric solutions of matrix Equation (4). Xie and Wang [31] deduced the reducible solution to quaternion matrix Equation (4). Additionally, Chen et al. [32] determined the necessary and sufficient conditions for the solvability of dual quaternion matrix Equation (4), and further provided the expression for the general solution when it is solvable.

Until now, there has been a scarcity of knowledge regarding matrix Equation (4) over the dual split quaternion algebra. Drawing inspiration from the preceding studies, this paper is dedicated to presenting the solvability conditions and providing the expression of the general solution for dual split quaternion matrix Equation (4).

This paper is organized as follows. In Section 2, we provide several basic definitions and properties that will serve as the foundation for our subsequent discussions in the following sections. In Section 3, we consider the necessary and sufficient conditions for solvability and the expression for the general solution regarding dual split quaternion matrix Equation (4). We also deduce the necessary and sufficient condition for the existence of the Hermitian solution to (4), and consider some particular instances of dual split quaternion matrix Equation (4). At the end, a numerical example is given in Section 4.

Throughout this paper, the sets of dual numbers, dual quaternions, and dual split quaternions are denoted by $\mathbb{D}$, $\mathbb{DH}$, and $\mathbb{DH}_s$, respectively. The sets of all $m \times n$ matrices over $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{H}_s$, $\mathbb{DH}$, and $\mathbb{DH}_s$ are denoted by $\mathbb{R}^{m\times n}$, $\mathbb{C}^{m\times n}$, $\mathbb{H}^{m\times n}$, $\mathbb{H}_s^{m\times n}$, $\mathbb{DH}^{m\times n}$, and $\mathbb{DH}_s^{m\times n}$, respectively. The symbols $I_n$, 0, and $A^*$ represent the $n \times n$ identity matrix, the zero matrix with appropriate size, and the conjugate transpose of $A$, respectively. $A^T$ and $A^\dagger$ denote the transpose and the Moore–Penrose inverse of matrix $A$, respectively. $L_A = I - A^\dagger A$ and $R_A = I - AA^\dagger$ are the two projectors induced by $A^\dagger$.

## 2. Preliminary

In this section, we explore the definitions of dual numbers, dual split quaternions, and associated properties. Additionally, we introduce the concept of dual split quaternion matrices and elaborate on the real representation for split quaternion matrices, which plays a pivotal role in the derivation of our main results.

### *2.1. Dual Numbers and Dual Split Quaternions*

The set of dual numbers is denoted by

$$\mathbb{D} = \{x = x_0 + x_1\epsilon | \epsilon^2 = 0, x_0, x_1 \in \mathbb{R}\},$$

where $\epsilon$ is the infinitesimal unit. We call $x_0$ the real part or the standard part of $x$, while $x_1$ is the dual part or the infinitesimal part of $x$. For any dual numbers $x = x = x_0 + x_1\epsilon$ and $y = y_0 + y_1\epsilon$, we have $x = y$ if $x_0 = y_0$ and $x_1 = y_1$, and the sum and product of $x$ and $y$ are defined as

$$x + y = x_0 + y_0 + (x_1 + y_1)\epsilon,$$
$$xy = x_0y_0 + (x_0y_1 + x_1y_0)\epsilon.$$

Moreover, the conjugate and norm of $x$ are defined by

$$\overline{x} = x_0 - x_1\epsilon,$$
$$r = |x| = \sqrt{x\overline{x}} = |x_0|,$$

respectively. The set of dual quaternions, which can be regarded as an extension of quaternions by incorporating dual numbers, is denoted as

$$\mathbb{DH} = \{q = q_0 + q_1 i + q_2 j + q_3 k : q_0, q_1, q_2, q_3 \in \mathbb{D}\},$$

where

$$i^2 = j^2 = k^2 = -1, ij = -ji = k, jk = -kj = i, ki = ik = j,$$

and

$$\epsilon i = i\epsilon, \epsilon j = j\epsilon, \epsilon k = k\epsilon, \epsilon \neq 0, \epsilon^2 = 0.$$

In a similar way, we can present the definition of a dual split quaternion, which can be regarded as an extension of split quaternions by incorporating dual numbers, is denoted as

$$\mathbb{DH}_s = \{q = q_0 + q_1 i + q_2 j + q_3 k : q_0, q_1, q_2, q_3 \in \mathbb{D}\},$$

where

$$i^2 = -j^2 = -k^2 = -1, ij = -ji = k, jk = -kj = -i, ki = -ik = j,$$

and

$$\epsilon i = i\epsilon, \epsilon j = j\epsilon, \epsilon k = k\epsilon, \epsilon \neq 0, \epsilon^2 = 0.$$

Now, we present the definitions of a quaternion matrix and dual split quaternion matrix, along with several definitions that are pertinent to our discussion.

Let $X_0, X_1 \in \mathbb{H}^{m\times n}(\mathbb{H}_s^{m\times n})$. $X$ is said to be a dual quaternion (dual split quaternion) matrix if $X$ takes the form $X = X_0 + X_1\epsilon$, where the set of all dual quaternion matrices and all dual split quaternion matrices are denoted by

$$\mathbb{DH}^{m\times n} = \{X = X_0 + X_1\epsilon | \epsilon^2 = 0, X_0, X_1 \in \mathbb{H}^{m\times n}\},$$

and

$$\mathbb{DH}_s^{m\times n} = \{X = X_0 + X_1\epsilon | \epsilon^2 = 0, X_0, X_1 \in \mathbb{H}_s^{m\times n}\},$$

respectively.

The set of $n \times n$ dual split quaternion matrices, which are equipped with standard matrix summation and multiplication operations, constitutes a ring with unity. Given any matrix $A = (A_{ij}) \in \mathbb{DH}_s^{m\times n}$ and $q \in \mathbb{DH}_s$, right and left scalar multiplications are defined as $Aq = (A_{ij}q)$ and $qA = (qA_{ij})$, respectively. Consequently, $\mathbb{DH}_s^{m\times n}$ is a left (right) vector space over $\mathbb{DH}_s$. Given any matrix $A = A_0 + A_1\epsilon = (A_{ij}) \in \mathbb{DH}_s^{m\times n}$, the Hamiltonian conjugate of $A$ is defined as $\overline{A} = \overline{A_0} + \overline{A_1}\epsilon = (\overline{A_{ij}}) \in \mathbb{DH}_s^{m\times n}$ , the transpose of $A$ is given by $A^T = {A_0}^T + {A_1}^T\epsilon = (A_{ji}) \in \mathbb{DH}_s^{n\times m}$, and the conjugate transpose of $A$ is defined as $A^* = {A_0}^* + {A_1}^*\epsilon = (\overline{A})^T \in \mathbb{DH}_s^{n\times m}$.

### 2.2. Real Representation of Split Quaternion Matrices and Its Properties

For any matrix $A \in {\mathbb{H}_s}^{m\times n}$, it can be uniquely represented as $A = A_1 + A_2 i + A_3 j + A_4 k$, where $A_1, A_2, A_3, A_4 \in \mathbb{R}^{m\times n}$, and $A^* = {A_1}^T - {A_2}^T i - {A_3}^T j - {A_4}^T k$ is the usual conjugate transpose of $A$. In addition, we define the $i$-conjugate and $i$-conjugate transpose as follows:

$$A^i = i^{-1}Ai = A_1 + A_2 i - A_3 j - A_4 k,$$
$$A^{i*} = -iA^* i = {A_1}^T - {A_2}^T i + {A_3}^T j + {A_4}^T k.$$

It is evident that $A^{i*} = (A^*)^i = (A^i)^*$.

The real representation method is crucial in analyzing the foundational theory of split quaternions. For $A \in \mathbb{H}_s^{m\times n}$, $A = A_1 + A_2 i + A_3 j + A_4 k$, where $A_1, A_2, A_3, A_4 \in \mathbb{R}^{m\times n}$, we define

$$A^{\sigma_1} := \begin{pmatrix} A_1 & A_2 & A_3 & A_4 \\ -A_2 & A_1 & -A_4 & A_3 \\ A_3 & -A_4 & A_1 & -A_2 \\ A_4 & A_3 & A_2 & A_1 \end{pmatrix}.$$

To further explore the properties of split quaternion matrices, based on the classical real representation $A^{\sigma_1}$, we define a new real representation as follows.

**Definition 1.** *Suppose that $A = A_1 + A_2 i + A_3 j + A_4 k \in \mathbb{H}_s^{m\times n}$, where $A_1, A_2, A_3, A_4 \in \mathbb{R}^{m\times n}$. We define*

$$A^{\sigma_i} := U_m A^{\sigma_1} = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 \\ -A_2 & A_1 & -A_4 & A_3 \\ -A_3 & A_4 & -A_1 & A_2 \\ -A_4 & -A_3 & -A_2 & -A_1 \end{pmatrix}, U_m = \begin{pmatrix} I_m & 0 & 0 & 0 \\ 0 & I_m & 0 & 0 \\ 0 & 0 & -I_m & 0 \\ 0 & 0 & 0 & -I_m \end{pmatrix}.$$

The properties of the real representations are presented subsequently. For simplicity, we denote

$$P_m = \begin{pmatrix} 0 & 0 & I_m & 0 \\ 0 & 0 & 0 & -I_m \\ I_m & 0 & 0 & 0 \\ 0 & -I_m & 0 & 0 \end{pmatrix}, Q_m = \begin{pmatrix} 0 & -I_m & 0 & 0 \\ I_m & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_m \\ 0 & 0 & I_m & 0 \end{pmatrix}, R_m = \begin{pmatrix} 0 & 0 & 0 & I_m \\ 0 & 0 & I_m & 0 \\ 0 & I_m & 0 & 0 \\ I_m & 0 & 0 & 0 \end{pmatrix}. \tag{5}$$

**Proposition 1.** *Let $A, B \in \mathbb{H}_s^{m\times n}$, $C \in \mathbb{H}_s^{n\times p}$, and $b \in \mathbb{R}$. Then,*

1. $A = B \Leftrightarrow A^{\sigma_1} = B^{\sigma_1}, A = B \Leftrightarrow A^{\sigma_i} = B^{\sigma_i}$.
2. $(A+B)^{\sigma_1} = A^{\sigma_1} + B^{\sigma_1}$ *and* $(bA)^{\sigma_1} = bA^{\sigma_1}$; $(A+B)^{\sigma_i} = A^{\sigma_i} + B^{\sigma_i}$ *and* $(bA)^{\sigma_i} = bA^{\sigma_i}$.
3. $(AC)^{\sigma_1} = A^{\sigma_1}C^{\sigma_1}$ *and* $(AC)^{\sigma_i} = A^{\sigma_i}U_n C^{\sigma_i}$.
4. (i) ${P_m}^T A^{\sigma_1} P_n = A^{\sigma_1}, {Q_m}^T A^{\sigma_1} Q_n = A^{\sigma_1}, {R_m}^T A^{\sigma_1} R_n = A^{\sigma_1}$.
   (ii) ${P_m}^T A^{\sigma_i} P_n = -A^{\sigma_i}, {Q_m}^T A^{\sigma_i} Q_n = A^{\sigma_i}, {R_m}^T A^{\sigma_i} R_n = -A^{\sigma_i}$.
5. (i) $A = \frac{1}{2}\begin{pmatrix} I_m & I_m i & I_m j & I_m k \end{pmatrix} A^{\sigma_1} \begin{pmatrix} I_n \\ I_n i \\ I_n j \\ I_n k \end{pmatrix}$.
   (ii) $A = \frac{1}{2}\begin{pmatrix} I_m & I_m i & I_m j & I_m k \end{pmatrix} A^{\sigma_i} \begin{pmatrix} -I_n \\ -I_n i \\ -I_n j \\ -I_n k \end{pmatrix}$.
6. $(A^*)^{\sigma_i} = (A^{\sigma_i})^T$.
7. $(A^i)^{\sigma_i} = U_m A^{\sigma_i} U_n$.

The proof for Proposition 1 is relatively straightforward, and thus, we omit it.

## 3. The Solution of Matrix Equation (4)

In this section, we pay attention to deriving the solution to dual split quaternion matrix Equation (4). We start with several useful results over $\mathbb{H}$ or $\mathbb{DH}$, which also hold over $\mathbb{R}$.

**Lemma 1** ([27]). *Assume that $A$, $B$, and $C$ are given matrices with the appropriate dimensions over $\mathbb{H}$; then, quaternion matrix Equation (4) is consistent if and only if the following conditions are satisfied:*

$$R_A C = 0, \quad CL_B = 0.$$

*In this case, the general solution can be expressed as*

$$X = A^{\dagger}CB^{\dagger} + L_A U + V R_B,$$

*where $U$ and $V$ are any matrices over $\mathbb{H}$ with appropriate dimensions.*

**Lemma 2** ([31]). *Let $A_1, A_2, B_1, B_2$, and $C_1$ be given matrices with appropriate sizes. Set*

$$A = R_{A_1}C,\ B = B_1 L_{B_2},\ M = R_{A_1}A_2,\ C_1 = CL_{B_2}.$$

*Then, the following descriptions are equivalent:*

(1) *The quaternion matrix equation*

$$A_1X_1B_1 + A_1X_2B_2 + A_2X_3B_2 = C \tag{6}$$

*is consistent.*

(2) $R_M A = 0,\ R_{A_1}CL_{B_2} = 0,$ *and* $C_1 L_B = 0.$

(3)

$$r\begin{pmatrix} A_1 & A_2 & C \end{pmatrix} = r\begin{pmatrix} A_1 & A_2 \end{pmatrix},$$
$$r\begin{pmatrix} B_2 & 0 \\ C & A_1 \end{pmatrix} = r(B_2) + r(A_1),$$
$$r\begin{pmatrix} C_1 \\ B_1 \\ B_2 \end{pmatrix} = r\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}.$$

*In this case, the general solution to (6) can be expressed as follows:*

$$\begin{aligned} X_1 &= A_1^{\dagger}C_1B^{\dagger} + L_{A_1}V_1 + V_2R_B, \\ X_2 &= A_1^{\dagger}(C - A_1X_1B_1 - A_2X_3B_2)B_2^{\dagger} + T_1R_{B_2} + L_{A_1}T_2, \\ X_3 &= M^{\dagger}AB_2^{\dagger} + L_MU_1 + U_2R_{B_2}, \end{aligned}$$

*where $U_1, U_2, V_1, V_2, T_1$, and $T_2$ are arbitrary matrices over $\mathbb{H}$ with appropriate sizes.*

**Lemma 3** ([32]). *Let $A = A_0 + A_1\epsilon \in \mathbb{DH}^{m\times n}$, $B = B_0 + B_1\epsilon \in \mathbb{DH}^{r\times l}$, and $C = C_0 + C_1\epsilon \in \mathbb{DH}^{m\times l}$. Put*

$$A_2 = A_1L_{A_0},\ B_2 = R_{B_0}B_1,\ C_{11} = A_0A_0^{\dagger}C_0B_0^{\dagger}B_1,$$
$$C_{22} = A_1A_0^{\dagger}C_0B_0^{\dagger}B_0,\ C_2 = C_1 - C_{11} - C_{22},$$
$$M = R_{A_0}A_2,\ N = R_{A_0}C_2,\ E = B_2L_{B_0},\ F = C_2L_{B_0}.$$

*Then, the following statements are equivalent:*

(1) *Dual quaternion matrix Equation (4) is consistent.*

(2)

$$R_{A_0}C_0 = 0,\ C_0L_{B_0} = 0,$$
$$R_MN = 0,\ R_{A_0}C_2L_{B_0} = 0,\ FL_E = 0.$$

(3)

$$
\begin{aligned}
&r\begin{pmatrix} A_0 & C_0 \end{pmatrix} = r(A_0), r\begin{pmatrix} B_0 \\ C_0 \end{pmatrix} = r(B_0), \\
&r\begin{pmatrix} A_1 & A_0 & C_1 \\ A_0 & 0 & C_0 \end{pmatrix} = r\begin{pmatrix} A_1 & A_0 \\ A_0 & 0 \end{pmatrix}, \\
&r\begin{pmatrix} C_1 & A_0 \\ B_0 & 0 \end{pmatrix} = r(A_0) + r(B_0), \\
&r\begin{pmatrix} B_1 & B_0 \\ B_0 & 0 \\ C_1 & C_0 \end{pmatrix} = r\begin{pmatrix} B_1 & B_0 \\ B_0 & 0 \end{pmatrix}.
\end{aligned}
$$

*In this case, the general solution X of dual quaternion matrix Equation (4) can be expressed as* $X = X_0 + X_1\epsilon$, *where*

$$
\begin{aligned}
X_0 &= A_0^{\dagger} C_0 B_0^{\dagger} + L_{A_0} U + V R_{B_0}, \\
X_1 &= A_0^{\dagger}(C_2 - A_0 V B_2 - A_2 U B_0) B_0^{\dagger} + W_1 R_{B_0} + L_{A_0} W_2, \\
U &= M^{\dagger} N B_0^{\dagger} + L_M Q_1 + Q_2 R_{B_0}, \\
V &= A_0^{\dagger} F E^{\dagger} + L_{A_0} Q_3 + Q_4 R_E,
\end{aligned}
\tag{7}
$$

*Moreover,* $Q_i(i = \overline{1,4})$ *and* $W_i(i = \overline{1,2})$ *are arbitrary matrices over* $\mathbb{H}$ *with appropriate dimensions.*

Using the above lemmas and applying the real representation method of split quaternions, we can deduce the general solution of matrix Equation (4) over the dual split quaternion algebra.

**Theorem 1.** *Let* $A = A_{00} + A_{01}\epsilon \in \mathbb{DH}_s^{m\times n}$, $B = B_{00} + B_{01}\epsilon \in \mathbb{DH}_s^{r\times l}$, *and* $C = C_{00} + C_{01}\epsilon \in \mathbb{DH}_s^{m\times l}$. *Let*

$$A_0 = A_{00}{}^{\sigma_1}, A_1 = A_{01}{}^{\sigma_1}, B_0 = B_{00}{}^{\sigma_1}, B_1 = B_{01}{}^{\sigma_1}, C_0 = C_{00}{}^{\sigma_1}, C_1 = C_{01}{}^{\sigma_1}, \tag{8}$$

$$A_2 = A_1 L_{A_0},\ B_2 = R_{B_0} B_1,\ C_{11} = A_0 A_0^{\dagger} C_0 B_0^{\dagger} B_1, \tag{9}$$

$$C_{22} = A_1 A_0^{\dagger} C_0 B_0^{\dagger} B_0,\ C_2 = C_1 - C_{11} - C_{22}, \tag{10}$$

$$M = R_{A_0} A_2,\ N = R_{A_0} C_2,\ E = B_2 L_{B_0},\ F = C_2 L_{B_0}. \tag{11}$$

*Then, the following statements are equivalent:*

(1) *Dual split quaternion matrix Equation (4) is consistent.*

(2) *The system of real matrix equations*

$$
\begin{cases}
A_0 X_0 B_0 = C_0, \\
A_0 X_0 B_1 + A_0 X_1 B_0 + A_1 X_0 B_0 = C_1,
\end{cases}
\tag{12}
$$

*is consistent.*

(3)

$$R_{A_0} C_0 = 0,\ C_0 L_{B_0} = 0, \tag{13}$$

$$R_M N = 0,\ R_{A_0} C_2 L_{B_0} = 0,\ F L_E = 0. \tag{14}$$

*(4)*

$$r\begin{pmatrix} A_0 & C_0 \end{pmatrix} = r(A_0), r\begin{pmatrix} B_0 \\ C_0 \end{pmatrix} = r(B_0), \tag{15}$$

$$r\begin{pmatrix} A_1 & A_0 & C_1 \\ A_0 & 0 & C_0 \end{pmatrix} = r\begin{pmatrix} A_1 & A_0 \\ A_0 & 0 \end{pmatrix}, \tag{16}$$

$$r\begin{pmatrix} C_1 & A_0 \\ B_0 & 0 \end{pmatrix} = r(A_0) + r(B_0), \tag{17}$$

$$r\begin{pmatrix} B_1 & B_0 \\ B_0 & 0 \\ C_1 & C_0 \end{pmatrix} = r\begin{pmatrix} B_1 & B_0 \\ B_0 & 0 \end{pmatrix}. \tag{18}$$

*In this case, the general solution $X$ of dual split quaternion matrix Equation (4) can be expressed as $X = X_{00} + X_{01}\epsilon$, where*

$$\begin{aligned} X_{00} &= \frac{1}{8}\begin{pmatrix} I_n & I_n i & I_n j & I_n k \end{pmatrix}(X_0 + P_n X_0 P_r{}^T + Q_n X_0 Q_r{}^T + R_n X_0 R_r{}^T)\begin{pmatrix} I_r \\ I_r i \\ I_r j \\ I_r k \end{pmatrix}, \\ X_{01} &= \frac{1}{8}\begin{pmatrix} I_n & I_n i & I_n j & I_n k \end{pmatrix}(X_1 + P_n X_1 P_r{}^T + Q_n X_1 Q_r{}^T + R_n X_1 R_r{}^T)\begin{pmatrix} I_r \\ I_r i \\ I_r j \\ I_r k \end{pmatrix}, \end{aligned} \tag{19}$$

*where*

$$\begin{aligned} X_0 &= A_0^\dagger C_0 B_0^\dagger + L_{A_0} U + V R_{B_0}, \\ X_1 &= A_0^\dagger (C_2 - A_0 V B_2 - A_2 U B_0) B_0^\dagger + W_1 R_{B_0} + L_{A_0} W_2, \\ U &= M^\dagger N B_0^\dagger + L_M Q_1 + Q_2 R_{B_0}, \\ V &= A_0^\dagger F E^\dagger + L_{A_0} Q_3 + Q_4 R_E, \end{aligned} \tag{20}$$

*and $Q_i (i = \overline{1,4})$ and $W_i (i = \overline{1,2})$ are arbitrary matrices over $\mathbb{R}$ with appropriate dimensions.*

**Proof.** (1) $\Leftrightarrow$ (2): Assume that dual split quaternion matrix Equation (4) has a solution denoted as $X \in \mathbb{DH}_s^{n\times r}$, which can be expressed as

$$X = X_{00} + X_{01}\epsilon, \tag{21}$$

where $X_{00}, X_{01} \in \mathbb{H}_s^{n\times r}$. Let $X_0 = X_{00}{}^{\sigma_1}$ and $X_1 = X_{01}{}^{\sigma_1}$. By substituting (21) into (4) and utilizing the definition of equality for dual split quaternion matrices, we can obtain that dual split quaternion matrix Equation (4) is equivalent to the system of split quaternion matrix equations

$$\begin{cases} A_{00} X_{00} B_{00} = C_{00}, \\ A_{00} X_{00} B_{01} + A_{00} X_{01} B_{00} + A_{01} X_{00} B_{00} = C_{01}. \end{cases} \tag{22}$$

Applying (3) of Proposition 1 to (12) yields

$$\begin{cases} A_{00}{}^{\sigma_1} X_{00}{}^{\sigma_1} B_{00}{}^{\sigma_1} = C_{00}{}^{\sigma_1}, \\ A_{00}{}^{\sigma_1} X_{00}{}^{\sigma_1} B_{01}{}^{\sigma_1} + A_{00}{}^{\sigma_1} X_{01}{}^{\sigma_1} B_{00}{}^{\sigma_1} + A_{01}{}^{\sigma_1} X_{00}{}^{\sigma_1} B_{00}{}^{\sigma_1} = C_{01}{}^{\sigma_1}, \end{cases} \tag{23}$$

i.e.,

$$\begin{cases} A_0 X_0 B_0 = C_0, \\ A_0 X_0 B_1 + A_0 X_1 B_0 + A_1 X_0 B_0 = C_1. \end{cases}$$

Clearly, $(X_0, X_1)$ is a pair of solutions to the system (12).

Conversely, if the real system has a pair of solutions $(X_0, X_1)$, which can be expressed as

$$X_0 = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \in \mathbb{R}^{4n\times 4r},$$

and

$$X_1 = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{pmatrix} \in \mathbb{R}^{4n\times 4r},$$

respectively, where $a_{ij}, b_{ij} \in \mathbb{R}^{n\times r}, (i, j = \overline{1,2})$, then, using (4) of Proposition 1 to the above equations, we can obtain

$$\begin{cases} {P_m}^T A_0 P_n X_0 {P_r}^T B_0 P_l = {P_m}^T C_0 P_l, \\ {P_m}^T A_0 P_n X_0 {P_r}^T B_1 P_l + {P_m}^T A_0 P_n X_1 {P_r}^T B_0 P_l + {P_m}^T A_1 P_n X_0 {P_r}^T B_0 P_l = {P_m}^T C_1 P_l. \end{cases}$$

Hence,

$$\begin{cases} A_0 P_n X_0 {P_r}^T B_0 = C_0, \\ A_0 P_n X_0 {P_r}^T B_1 + A_0 P_n X_1 {P_r}^T B_0 + A_1 P_n X_0 {P_r}^T B_0 = C_1, \end{cases}$$

and it follows that $(P_n X_0 {P_r}^T, P_n X_1 {P_r}^T)$ is a pair of solutions to system (12). Similarly, $(Q_n X_0 {Q_r}^T$ and $Q_n X_1 {Q_r}^T)$, $(R_n X_0 {R_r}^T, R_n X_1 {R_r}^T)$ are also pairs of solutions to system (12). Then, so is $(Y_0, Y_1)$, where

$$Y_0 = \frac{1}{4}(X_0 + P_n X_0 {P_r}^T + Q_n X_0 {Q_r}^T + R_n X_0 {R_r}^T),$$

$$Y_1 = \frac{1}{4}(X_1 + P_n X_1 {P_r}^T + Q_n X_1 {Q_r}^T + R_n X_1 {R_r}^T).$$

By direct computation, we have

$$Y_0 = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ -c_2 & c_1 & -c_4 & c_3 \\ c_3 & -c_4 & c_1 & -c_2 \\ c_4 & c_3 & c_2 & c_1 \end{pmatrix},$$

$$Y_1 = \begin{pmatrix} d_1 & d_2 & d_3 & d_4 \\ -d_2 & d_1 & -d_4 & d_3 \\ d_3 & -d_4 & d_1 & -d_2 \\ d_4 & d_3 & d_2 & d_1 \end{pmatrix},$$

where

$$c_1 = \frac{1}{4}(a_{11} + a_{22} + a_{33} + a_{44}), \quad c_2 = \frac{1}{4}(a_{12} - a_{21} - a_{34} + a_{43}),$$

$$c_3 = \frac{1}{4}(a_{13} + a_{24} + a_{31} + a_{42}), \quad c_4 = \frac{1}{4}(a_{14} - a_{23} - a_{32} + a_{41}),$$

and

$$d_1 = \frac{1}{4}(b_{11} +_{22} + b_{33} + b_{44}), \quad d_2 = \frac{1}{4}(b_{12} - b_{21} - b_{34} + b_{43}),$$

$$d_3 = \frac{1}{4}(b_{13} + b_{24} + b_{31} + b_{42}), \quad d_4 = \frac{1}{4}(b_{14} - b_{23} - b_{32} + b_{41}).$$

Now, we obtain that

$$X_{00}=c_1+c_2i+c_3j+c_4k=\frac{1}{2}\begin{pmatrix}I_n & I_ni & I_nj & I_nk\end{pmatrix}Y_0\begin{pmatrix}I_r\\ I_ri\\ I_rj\\ I_rk\end{pmatrix}\in \mathbb{H}_s{}^{n\times r},$$

$$X_{01}=d_1+d_2i+d_3j+d_4k=\frac{1}{2}\begin{pmatrix}I_n & I_ni & I_nj & I_nk\end{pmatrix}Y_1\begin{pmatrix}I_r\\ I_ri\\ I_rj\\ I_rk\end{pmatrix}\in \mathbb{H}_s{}^{n\times r}.$$

According to (5) of Proposition 1, ${X_{00}}^{\sigma_1}=Y_0$ and ${X_{01}}^{\sigma_1}=Y_1$. Consequently,

$$\begin{cases}{A_{00}}^{\sigma_1}{X_{00}}^{\sigma_1}{B_{00}}^{\sigma_1}={C_{00}}^{\sigma_1},\\ {A_{00}}^{\sigma_1}{X_{00}}^{\sigma_1}{B_{01}}^{\sigma_1}+{A_{00}}^{\sigma_1}{X_{01}}^{\sigma_1}{B_{00}}^{\sigma_1}+{A_{01}}^{\sigma_1}{X_{00}}^{\sigma_1}{B_{00}}^{\sigma_1}={C_{01}}^{\sigma_1},\end{cases}$$

indicating that $(X_{00},X_{01})$ is a pair of solutions to the system of split quaternion matrix Equation (22). From Lemma 3, we can easily know that the system of the split quaternion matrix Equation (22) is equivalent to dual split quaternion matrix Equation (4). Thus, matrix Equation (4) has a dual split solution $X\in\mathbb{DH}_s^{n\times r}$ if and only if the system of real matrix Equation (12) is consistent. And in such a case, the general solution to dual split quaternion matrix Equation (4) can be expressed as (19) and (20).

According to Lemmas 1–3, we can easily verify that system (12) is consistent if and only if (13)–(18) hold. Thus, we have shown the equivalence of (2)–(4). □

As an application of the above theorem and real representation method, next we investigate the necessary and sufficient conditions for the existence of Hermitian solution to dual split quaternion matrix Equation (4).

**Theorem 2.** *Let $A=A_{00}+A_{01}\epsilon\in\mathbb{DH}_s^{m\times n}$, $B=B_{00}+B_{01}\epsilon\in\mathbb{DH}_s^{n\times l}$, and $C=C_{00}+C_{01}\epsilon\in\mathbb{DH}_s^{m\times l}$. Let*

$$A_0={A_{00}}^{\sigma_i}, A_1={A_{01}}^{\sigma_i},$$
$$B_0={B_{00}}^{\sigma_i}, B_1={B_{01}}^{\sigma_i},$$
$$C_0={C_{00}}^{\sigma_i}, C_1={C_{01}}^{\sigma_i}.$$

*Then, dual split quaternion matrix Equation (4) has a Hermitian solution $X=X^*\in\mathbb{DH}_s^{n\times n}$ if and only if the system of real matrix equations*

$$\begin{cases}A_0X_0B_0=C_0,\\ A_0X_0B_1+A_0X_1B_0+A_1X_0B_0=C_1,\end{cases}\tag{24}$$

*has a pair of symmetric solutions $(X_0,X_1)$.*

**Proof.** Assume that $X=X^*\in\mathbb{DH}_s^{n\times n}$ is a solution to dual split quaternion matrix Equation (4), which can be expressed as

$$X=X_{00}+X_{01}\epsilon,\tag{25}$$

where $X_{00},X_{01}\in\mathbb{H}_s^{n\times n}$, $X_{00}={X_{00}}^*$, and $X_{01}={X_{01}}^*$. Let $X_0=U_n{X_{00}}^{\sigma_i}U_n$ and $X_1=U_n{X_{01}}^{\sigma_i}U_n$. By combining (24) and (6) of Proposition 1, we can obtain that

$$\begin{cases}{A_{00}}^{\sigma_i}U_n{X_{00}}^{\sigma_i}U_n{B_{00}}^{\sigma_i}={C_{00}}^{\sigma_i},\\ {A_{00}}^{\sigma_i}U_n{X_{00}}^{\sigma_i}U_n{B_{01}}^{\sigma_i}+{A_{00}}^{\sigma_i}U_n{X_{01}}^{\sigma_i}U_n{B_{00}}^{\sigma_i}+{A_{01}}^{\sigma_i}U_n{X_{00}}^{\sigma_i}U_n{B_{00}}^{\sigma_i}={C_{01}}^{\sigma_i},\end{cases}\tag{26}$$

and

$$X_{00}^{\sigma_i} = (X_{00}{}^{*})^{\sigma_i} = (X_{00}{}^{\sigma_i})^T,$$
$$X_{01}^{\sigma_i} = (X_{01}{}^{*})^{\sigma_i} = (X_{01}{}^{\sigma_i})^T,$$

i.e.,

$$\begin{cases} A_0 X_0 B_0 = C_0, \\ A_0 X_0 B_1 + A_0 X_1 B_0 + A_1 X_0 B_0 = C_1, \end{cases}$$

and

$$X_0 = X_0{}^T, X_1 = X_1{}^T.$$

Conversely, if the system of real matrix Equation (24) has a pair of symmetric solutions $(X_0, X_1)$, which can be expressed as

$$X_0 = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \in \mathbb{R}^{4n \times 4n},$$

and

$$X_1 = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{pmatrix} \in \mathbb{R}^{4n \times 4n},$$

respectively, where $a_{ij}, b_{ij} \in \mathbb{R}^{n \times n}$ $(i, j = \overline{1, 2})$, then (24) holds, and

$$\begin{cases} A_{00}{}^{\sigma_i} X_0 B_{00}{}^{\sigma_i} = C_{00}{}^{\sigma_i}, \\ A_{00}{}^{\sigma_i} X_0 B_{01}{}^{\sigma_i} + A_{00}{}^{\sigma_i} X_1 B_{00}{}^{\sigma_i} + A_{01}{}^{\sigma_i} X_0 B_{00}{}^{\sigma_i} = C_{01}{}^{\sigma_i}, \end{cases} \tag{27}$$

where $X_0 = X_0{}^T$ and $X_1 = X_1{}^T$. According to (4) of Proposition 1, we can obtain that $(-P_n X_0 P_n{}^T, -P_n X_1 P_n{}^T)$, $(Q_n X_0 Q_n{}^T, Q_n X_1 Q_n{}^T)$, and $(-R_n X_0 R_n{}^T, -R_n X_1 R_n{}^T)$ are also pairs of symmetric solutions to system (24). Then, so is $(Y_0, Y_1)$, where

$$Y_0 = \frac{1}{4}(X_0 - P_n X_0 P_n{}^T + Q_n X_0 Q_n{}^T - R_n X_0 R_n{}^T),$$

$$Y_1 = \frac{1}{4}(X_1 - P_n X_1 P_n{}^T + Q_n X_1 Q_n{}^T - R_n X_1 R_n{}^T).$$

By direct computation, we have

$$Y_0 = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ -c_2 & c_1 & -c_4 & c_3 \\ -c_3 & c_4 & -c_1 & c_2 \\ -c_4 & -c_3 & -c_2 & -c_1 \end{pmatrix},$$

$$Y_1 = \begin{pmatrix} d_1 & d_2 & d_3 & d_4 \\ -d_2 & d_1 & -d_4 & d_3 \\ -d_3 & d_4 & -d_1 & d_2 \\ -d_4 & -d_3 & -d_2 & -d_1 \end{pmatrix},$$

where

$$c_1 = \frac{1}{4}(a_{11} + a_{22} - a_{33} - a_{44}), \quad c_2 = \frac{1}{4}(a_{12} - a_{21} + a_{34} - a_{43}),$$

$$c_3 = \frac{1}{4}(a_{13} + a_{24} - a_{31} - a_{42}), \quad c_4 = \frac{1}{4}(a_{14} - a_{23} + a_{32} - a_{41}),$$

and

$$d_1 = \frac{1}{4}(b_{11} +_{22} - b_{33} - b_{44}), \quad d_2 = \frac{1}{4}(b_{12} - b_{21} + b_{34} - b_{43}),$$
$$d_3 = \frac{1}{4}(b_{13} + b_{24} - b_{31} - b_{42}), \quad d_4 = \frac{1}{4}(b_{14} - b_{23} + b_{32} - b_{41}).$$

Now, we obtain that

$$X_{00} = c_1 + c_2 i + c_3 j + c_4 k = \frac{1}{2}\begin{pmatrix} I_n & I_n i & I_n j & I_n k \end{pmatrix} Y_0 \begin{pmatrix} -I_n \\ -I_n i \\ -I_n j \\ -I_n k \end{pmatrix} \in \mathbb{H}_s{}^{n\times n},$$

$$X_{01} = d_1 + d_2 i + d_3 j + d_4 k = \frac{1}{2}\begin{pmatrix} I_n & I_n i & I_n j & I_n k \end{pmatrix} Y_1 \begin{pmatrix} -I_n \\ -I_n i \\ -I_n j \\ -I_n k \end{pmatrix} \in \mathbb{H}_s{}^{n\times n}.$$

According to (5) of Proposition 1, ${X_{00}}^{\sigma_i} = Y_0$ and ${X_{01}}^{\sigma_i} = Y_1$. Consequently,

$$\begin{cases} {A_{00}}^{\sigma_i}{X_{00}}^{\sigma_i}{B_{00}}^{\sigma_i} = {C_{00}}^{\sigma_i}, \\ {A_{00}}^{\sigma_i}{X_{00}}^{\sigma_i}{B_{01}}^{\sigma_i} + {A_{00}}^{\sigma_i}{X_{01}}^{\sigma_i}{B_{00}}^{\sigma_i} + {A_{01}}^{\sigma_i}{X_{00}}^{\sigma_i}{B_{00}}^{\sigma_i} = {C_{01}}^{\sigma_i}, \end{cases}$$

indicating that $({X_{00}}^{\sigma_i},{X_{01}}^{\sigma_i})$ is a pair of symmetric solutions to system (24). From (7) of Proposition 1, we can easily obtain that $(U_n(X_{00}^i)^{\sigma_i}U_n, U_n(X_{01}^i)^{\sigma_i}U_n)$ is also a pair of symmetric solutions to system (24). Thus,

$$\begin{cases} {A_{00}}^{\sigma_i}U_n(X_{00}^i)^{\sigma_i}U_n{B_{00}}^{\sigma_i} = {C_{00}}^{\sigma_i}, \\ {A_{00}}^{\sigma_i}U_n(X_{00}^i)^{\sigma_i}U_n{B_{01}}^{\sigma_i} + {A_{00}}^{\sigma_i}U_n(X_{01}^i)^{\sigma_i}U_n{B_{00}}^{\sigma_i} + {A_{01}}^{\sigma_i}U_n(X_{00}^i)^{\sigma_i}U_n{B_{00}}^{\sigma_i} = {C_{01}}^{\sigma_i}, \end{cases}$$

and

$$(X_{00}^i)^{\sigma_i} = ((X_{00}^i)^{\sigma_i})^T = ((X_{00}^i)^*)^{\sigma_i},$$
$$(X_{01}^i)^{\sigma_i} = ((X_{01}^i)^{\sigma_i})^T = ((X_{01}^i)^*)^{\sigma_i},$$

i.e.,

$$\begin{cases} A_{00}X_{00}^i B_{00} = C_{00}, \\ A_{00}X_{00}^i B_{01} + A_{00}X_{01}^i B01_{00} + A_{01}X_{00}^i B_{00} = C_{01}. \end{cases} \tag{28}$$

and

$$X_{00}^i = (X_{00}^i)^*,$$
$$X_{01}^i = (X_{01}^i)^*,$$

which indicates that dual split quaternion matrix Equation (4) has a Hermitian solution $X = X_{00}^i + X_{01}^i\epsilon$. □

Now, let us turn our attention to some specific instances of dual split quaternion matrix Equation (4).

**Corollary 1.** *Let $A = A_{00} + A_{01}\epsilon \in \mathbb{DH}_s^{m\times n}$, $C = C_{00} + C_{01}\epsilon \in \mathbb{DH}_s^{m\times r}$ be known. Let*

$$A_0 = {A_{00}}^{\sigma_1},\ A_1 = {A_{01}}^{\sigma_1},\ C_0 = {C_{00}}^{\sigma_1},\ C_1 = {C_{01}}^{\sigma_1}, \tag{29}$$

$$A_2 = A_1 L_{A_0},\ C_{22} = A_1 A_0^{\dagger} C_0,\ C_2 = C_1 - C_{22},\ M = R_{A_0}A_2,\ N = R_{A_0}C_2. \tag{30}$$

*Then, the following statements are equivalent:*

*(1) The dual split quaternion matrix equation $AX = C$ is consistent.*

(2) *The system of real matrix equations*

$$\begin{cases} A_0X_0 = C_0, \\ A_0X_1 + A_1X_0 = C_1, \end{cases} \tag{31}$$

*is consistent.*

(3)

$$R_{A_0}C_0 = 0, R_M N = 0. \tag{32}$$

(4)

$$r\begin{pmatrix} A_0 & C_0 \end{pmatrix} = r(A_0), r\begin{pmatrix} A_1 & A_0 & C_1 \\ A_0 & 0 & C_0 \end{pmatrix} = r\begin{pmatrix} A_1 & A_0 \\ A_0 & 0 \end{pmatrix}. \tag{33}$$

*In this case, the general solution $X$ of the dual split quaternion matrix equation $AX = C$ can be expressed as $X = X_{00} + X_{01}\epsilon$, where*

$$\begin{aligned} X_{00} &= \frac{1}{8}\begin{pmatrix} I_n & I_n i & I_n j & I_n k \end{pmatrix}(X_0 + P_n X_0 {P_r}^T + Q_n X_0 {Q_r}^T + R_n X_0 {R_r}^T)\begin{pmatrix} I_r \\ I_r i \\ I_r j \\ I_r k \end{pmatrix}, \\ X_{01} &= \frac{1}{8}\begin{pmatrix} I_n & I_n i & I_n j & I_n k \end{pmatrix}(X_1 + P_n X_1 {P_r}^T + Q_n X_1 {Q_r}^T + R_n X_1 {R_r}^T)\begin{pmatrix} I_r \\ I_r i \\ I_r j \\ I_r k \end{pmatrix}, \end{aligned} \tag{34}$$

*where*

$$\begin{aligned} X_0 &= A_0^\dagger C_0 + L_{A_0}U, \\ X_1 &= A_0^\dagger (C_2 - A_2 U) + L_{A_0}W_1, \\ U &= M^\dagger N + L_M W_2, \end{aligned} \tag{35}$$

*and $W_1$ and $W_2$ are arbitrary matrices over $\mathbb{R}$ with appropriate dimensions.*

**Corollary 2.** *Let $B = B_0 + B_1\epsilon \in \mathbb{DH}_s^{r\times l}$ and $C = C_0 + C_1\epsilon \in \mathbb{DH}_s^{n\times l}$ be known. Denote*

$$B_0 = {B_{00}}^{\sigma_1},\ B_1 = {B_{01}}^{\sigma_1},\ C_0 = {C_{00}}^{\sigma_1},\ C_1 = {C_{01}}^{\sigma_1}, \tag{36}$$

$$B_2 = R_{B_0}B_1,\ C_{11} = C_0 B_0^\dagger B_1,\ C_2 = C_1 - C_{11},\ E = B_2 L_{B_0},\ F = C_2 L_{B_0}. \tag{37}$$

*Then, the following statements are equivalent:*

(1) *The dual split quaternion matrix equation $XB = C$ is consistent.*

(2) *The system of real matrix equations*

$$\begin{cases} X_0B_0 = C_0, \\ X_1B_0 + X_0B_1 = C_1, \end{cases} \tag{38}$$

*is consistent.*

(3)

$$C_0 L_{B_0} = 0,\ F L_E = 0. \tag{39}$$

(4)

$$r\begin{pmatrix} B_0 \\ C_0 \end{pmatrix} = r(B_0),\ r\begin{pmatrix} B_1 & B_0 \\ B_0 & 0 \\ C_1 & C_0 \end{pmatrix} = r\begin{pmatrix} B_1 & B_0 \\ B_0 & 0 \end{pmatrix}. \tag{40}$$

*In this case, the general solution $X$ of the dual split quaternion matrix equation $XB = C$ can be expressed as $X = X_{00} + X_{01}\epsilon$, where*

$$X_{00} = \frac{1}{8}\begin{pmatrix} I_n & I_n i & I_n j & I_n k \end{pmatrix}(X_0 + P_n X_0 {P_r}^T + Q_n X_0 {Q_r}^T + R_n X_0 {R_r}^T)\begin{pmatrix} I_r \\ I_r i \\ I_r j \\ I_r k \end{pmatrix},$$
$$X_{01} = \frac{1}{8}\begin{pmatrix} I_n & I_n i & I_n j & I_n k \end{pmatrix}(X_1 + P_n X_1 {P_r}^T + Q_n X_1 {Q_r}^T + R_n X_1 {R_r}^T)\begin{pmatrix} I_r \\ I_r i \\ I_r j \\ I_r k \end{pmatrix}, \tag{41}$$

*where*

$$\begin{aligned} X_0 &= C_0 B_0^{\dagger} + V R_{B_0}, \\ X_1 &= (C_2 - V B_2) B_0^{\dagger} + W_1 R_{B_0}, \\ V &= F E^{\dagger} + W_2 R_E, \end{aligned} \tag{42}$$

*and $W_1$ and $W_2$ are arbitrary matrices over $\mathbb{R}$ with appropriate dimensions.*

## 4. Numerical Example

Now, we provide a numerical example to further clarify the main findings of this paper.

$$\begin{aligned} A &= A_{00} + A_{01}\epsilon = \begin{pmatrix} i+k & i+j \\ j & i-k \end{pmatrix} + \begin{pmatrix} 2+j-k & i-j \\ 0 & 4+4i \end{pmatrix}\epsilon, \\ B &= B_{00} + B_{01}\epsilon = \begin{pmatrix} 1+i+j & i-k \\ 0 & j+k \end{pmatrix} + \begin{pmatrix} i-k & 3k \\ i+j & 3i+k \end{pmatrix}\epsilon, \\ C &= C_{00} + C_{01}\epsilon \\ &= \begin{pmatrix} -4-7i-5j-6k & 6-8i-14j-2k \\ -10-8i-14j & -7-13i-14j+6k \end{pmatrix} \\ &+ \begin{pmatrix} -2+3i-10i+21k & -50-45i-17j-12k \\ -35-4i-29j+37k & -67+i-43j+31k. \end{pmatrix}\epsilon. \end{aligned}$$

From MATLAB9.10, we obtain

$$A_0 = {A_{00}}^{\sigma_1} = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 \\ -1 & -1 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & -1 & -1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

$$A_1 = {A_{01}}^{\sigma_1} = \begin{pmatrix} 2 & 0 & 0 & 1 & 1 & -1 & -1 & 0 \\ 0 & 4 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & 1 & 0 & 1 & -1 \\ 0 & -4 & 0 & 4 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 2 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & -4 \\ -1 & 0 & 1 & -1 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & 4 \end{pmatrix},$$

$$B_0 = {B_{00}}^{\sigma_1} = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ -1 & -1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B_1 = {B_{01}}^{\sigma_1} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & -1 & 3 \\ 0 & 0 & 1 & 3 & 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 1 & -3 & 0 & 0 \\ -1 & -3 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & -1 & -3 \\ -1 & 3 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 3 & 0 & 0 \end{pmatrix},$$

$$C_0 = {C_{00}}^{\sigma_1} = \begin{pmatrix} -4 & 6 & -7 & -8 & -5 & -14 & -6 & -2 \\ -10 & -7 & -8 & -13 & -14 & -14 & 0 & 6 \\ 7 & 8 & -4 & 6 & 6 & 2 & -5 & -14 \\ 8 & 13 & -10 & -7 & 0 & -6 & -14 & -14 \\ -5 & -14 & 6 & 2 & -4 & 6 & 7 & 8 \\ -14 & -14 & 0 & -6 & -10 & -7 & 8 & 13 \\ -6 & -2 & -5 & -14 & -7 & -8 & -4 & 6 \\ 0 & 6 & -14 & -14 & -8 & -13 & -10 & -7 \end{pmatrix},$$

$$C_1 = {C_{01}}^{\sigma_1} = \begin{pmatrix} -2 & -50 & 3 & -45 & -10 & -17 & 21 & -12 \\ -35 & -67 & -4 & 1 & -29 & -43 & 37 & 31 \\ -3 & 45 & -2 & -50 & -21 & 12 & -10 & -17 \\ 4 & -1 & -35 & -67 & -37 & -31 & -29 & -43 \\ -10 & -17 & -21 & 12 & -2 & -50 & -3 & 45 \\ -29 & -43 & -37 & -31 & -35 & -67 & 4 & -1 \\ 21 & -12 & -10 & -17 & 3 & -45 & -2 & -50 \\ 37 & 31 & -29 & -43 & -4 & 1 & -35 & -67 \end{pmatrix},$$

and

$$\begin{aligned} & r\begin{pmatrix} A_0 & C_0 \end{pmatrix} = r(A_0) = 8, \\ & r\begin{pmatrix} B_0 \\ C_0 \end{pmatrix} = r(B_0) = 8, \\ & r\begin{pmatrix} A_1 & A_0 & C_1 \\ A_0 & 0 & C_0 \end{pmatrix} = r\begin{pmatrix} A_1 & A_0 \\ A_0 & 0 \end{pmatrix} = 16, \\ & r\begin{pmatrix} C_1 & A_0 \\ B_0 & 0 \end{pmatrix} = r(A_0) + r(B_0) = 16, \\ & r\begin{pmatrix} B_1 & B_0 \\ B_0 & 0 \\ C_1 & C_0 \end{pmatrix} = r\begin{pmatrix} B_1 & B_0 \\ B_0 & 0 \end{pmatrix} = 16. \end{aligned}$$

Therefore, dual split quaternion matrix Equation (4) is consistent, and the general solution $X$ can be expressed as

$$X = X_{00} + X_{01}\epsilon,$$

where

$$X_{00} = \frac{1}{8}\begin{pmatrix} I_2 & I_2 i & I_2 j & I_2 k \end{pmatrix}(X_0 + P_2 X_0 {P_2}^T + Q_2 X_0 {Q_2}^T + R_2 X_0 {R_2}^T)\begin{pmatrix} I_2 \\ I_2 i \\ I_2 j \\ I_2 k \end{pmatrix},$$

$$X_{01} = \frac{1}{8}\begin{pmatrix} I_2 & I_2 i & I_2 j & I_2 k \end{pmatrix}(X_1 + P_2 X_1 {P_2}^T + Q_2 X_1 {Q_2}^T + R_2 X_1 {R_2}^T)\begin{pmatrix} I_2 \\ I_2 i \\ I_2 j \\ I_2 k \end{pmatrix},$$

where

$$\begin{aligned} X_0 &= \begin{pmatrix} 2.0127\times10^{-15} & 3 & 3 & 4 & -3.5933\times10^{-15} & 1.1003\times10^{-14} & 1 & 3.9968\times10^{-15} \\ 3 & 4 & 4 & 4 & -7.1979\times10^{-15} & 2.3148\times10^{-14} & -2.3662\times10^{-15} & 2 \\ -3 & -4 & 3.2837\times10^{-15} & 3 & -1 & -8.0045\times10^{-15} & -2.0435\times10^{-15} & 4.4409\times10^{-16} \\ -4 & -4 & 3 & 4 & 7.8752\times10^{-15} & -2 & 1.6862\times10^{-15} & 3.5527\times10^{-15} \\ 6.5817\times10^{-16} & 2.2204\times10^{-15} & -1 & -1.5321\times10^{-14} & 5.3291\times10^{-15} & 3 & -3 & -4 \\ -3.9526\times10^{-15} & -3.8858\times10^{-15} & 1.4846\times10^{-15} & -2 & 3 & 4 & -4 & 3 \\ 1 & 4.885\times10^{-15} & 4.8737\times10^{-15} & 2.4425\times10^{-15} & 3 & 4 & -4.8121\times10^{-15} & 3 \\ 6.6578\times10^{-15} & 2 & 2.1038\times10^{-15} & 1.0547\times10^{-14} & 4 & 4 & 3 & 4 \end{pmatrix} \\ &\quad + L_{A_0}U + VR_{B_0}, \\ X_1 &= A_0^{\dagger}(C_2 - A_0VB_2 - A_2UB_0)B_0^{\dagger} + W_1R_{B_0} + L_{A_0}W_2 \\ &= A_0^{\dagger}C_2B_0^{\dagger} - A_0^{\dagger}A_0VB_2B_0^{\dagger} - A_0^{\dagger}A_2UB_0B_0^{\dagger} + W_1R_{B_0} + L_{A_0}W_2, \\ U &= M^{\dagger}NB_0^{\dagger} + L_MQ_1 + Q_2R_{B_0}, \\ V &= A_0^{\dagger}FE^{\dagger} + L_{A_0}Q_3 + Q_4R_E, \end{aligned}$$

where

$$L_{A_0}=\begin{pmatrix} -2.2204e-16 & 2.3592e-16 & 1.3878e-17 & -2.7756e-17 & 3.6082e-16 & 2.0817e-16 & 1.7019e-17 & -5.5511e-17 \\ -3.0531e-16 & 0 & -4.4409e-16 & 8.3267e-17 & 5.5511e-17 & 5.5511e-17 & 5.5511e-17 & -1.6653e-16 \\ -5.5511e-17 & -5.5511e-17 & 0 & 1.6653e-16 & 5.5511e-17 & 3.3307e-16 & -1.1102e-16 & -4.9906e-17 \\ -1.1102e-16 & 2.2204e-16 & -1.6653e-16 & -2.2204e-16 & 2.7756e-16 & 5.5511e-17 & -1.1102e-16 & -2.2204e-16 \\ 1.6653e-16 & -1.5266e-16 & 1.249e-16 & 2.7756e-16 & -2.2204e-16 & -2.9143e-16 & -1.6677e-16 & -3.6082e-16 \\ -1.6653e-16 & -1.1102e-16 & 0 & 1.1102e-16 & -1.6653e-16 & 0 & 3.6082e-16 & 1.1102e-16 \\ -5.5511e-17 & -1.1102e-16 & 0 & -5.5511e-17 & 1.3878e-16 & 1.1102e-16 & -4.4409e-16 & 4.1525e-17 \\ -1.1102e-16 & -1.9429e-16 & -2.7756e-16 & -1.1102e-16 & -2.2204e-16 & -2.2204e-16 & 0 & -2.2204e-16 \end{pmatrix},$$

$$R_{B_0}=\begin{pmatrix} -4.4409e-16 & -5.5511e-16 & -1.3922e-16 & 1.4988e-15 & 4.7001e-16 & -1.5765e-15 & 1.0547e-15 & 9.992e-16 \\ -4.8392e-16 & 6.6613e-16 & 5.6769e-16 & 7.7716e-16 & 9.948e-16 & -1.5089e-15 & -8.5695e-16 & -7.6328e-16 \\ -8.5258e-17 & 3.8858e-16 & -2.2204e-16 & 1.4433e-15 & 0 & -2.2204e-15 & 4.1633e-17 & -1.6653e-16 \\ -4.7132e-16 & -3.8858e-16 & -2.4029e-16 & 8.8818e-16 & 2.8563e-16 & -1.0003e-15 & 6.4879e-16 & 2.0817e-16 \\ 4.4556e-16 & 3.3307e-16 & 2.8039e-16 & 2.2204e-16 & -2.2204e-16 & 1.4439e-16 & -2.7756e-16 & 1.1102e-16 \\ -9.4234e-16 & -4.5846e-16 & -7.8744e-16 & 4.2166e-16 & 2.3413e-16 & -1.3323e-15 & 9.194e-16 & 3.3307e-16 \\ -3.073e-16 & -4.996e-16 & 0 & -2.1094e-15 & 2.2204e-16 & 2.2204e-15 & 4.4409e-16 & 5.5511e-17 \\ -1.0272e-15 & -4.9803e-16 & 1.3749e-17 & -1.5751e-15 & 8.2236e-16 & 5.5511e-16 & 1.5613e-16 & -8.8818e-16 \end{pmatrix},$$

$$A_0^{\dagger}C_2B_0^{\dagger}=\begin{pmatrix} -15.5 & -18.5 & 5.5 & 10.5 & 9.5 & -1.5 & 12.5 & 15.5 \\ -9 & -3.5 & -0.5 & 16 & 14.5 & -2.5 & 12 & 6 \\ -5.5 & -10.5 & -15.5 & -18.5 & -12.5 & -15.5 & 9.5 & -1.5 \\ 0.5 & -16 & -9 & -3.5 & -12 & -6 & 14.5 & -2.5 \\ 9.5 & -1.5 & -12.5 & -15.5 & -15.5 & -18.5 & -5.5 & -10.5 \\ 14.5 & -2.5 & -12 & -6 & -9 & -3.5 & 0.5 & -16 \\ 12.5 & 15.5 & 9.5 & -1.5 & 5.5 & 10.5 & -15.5 & -18.5 \\ 12 & 6 & 14.5 & -2.5 & -0.5 & 16 & -9 & -3.5 \end{pmatrix},$$

$$A_0^{\dagger}A_0=\begin{pmatrix} 1 & -2.3592e-16 & -1.3878e-17 & 2.7756e-17 & -3.6082e-16 & -2.0817e-16 & -1.7019e-17 & 5.5511e-17 \\ 3.0531e-16 & 1 & 4.4409e-16 & -8.3267e-17 & -5.5511e-17 & -5.5511e-17 & -5.5511e-17 & 1.6653e-16 \\ 5.5511e-17 & 5.5511e-17 & 1 & -1.6653e-16 & -5.5511e-17 & -3.3307e-16 & 1.1102e-16 & 4.9906e-17 \\ 1.1102e-16 & -2.2204e-16 & 1.6653e-16 & 1 & -2.7756e-16 & -5.5511e-17 & 1.1102e-16 & 2.2204e-16 \\ -1.6653e-16 & 1.5266e-16 & -1.249e-16 & -2.7756e-16 & 1 & 2.9143e-16 & 1.6677e-16 & 3.6082e-16 \\ 1.6653e-16 & 1.1102e-16 & 0 & -1.1102e-16 & 1.6653e-16 & 1 & -3.6082e-16 & -1.1102e-16 \\ 5.5511e-17 & 1.1102e-16 & 0 & 5.5511e-17 & -1.3878e-16 & -1.1102e-16 & 1 & -4.1525e-17 \\ 1.1102e-16 & 1.9429e-16 & 2.7756e-16 & 1.1102e-16 & 2.2204e-16 & 2.2204e-16 & 0 & 1 \end{pmatrix},$$

$$B_2B_0^{\dagger}=\begin{pmatrix} -4.8803e-15 & -1.5856e-15 & 1.4114e-15 & 1.8483e-15 & 2.301e-15 & 1.9684e-15 & 3.9389e-15 & 3.0488e-15 \\ -1.1967e-15 & -2.0004e-15 & 6.3534e-16 & 4.38e-15 & -1.6479e-16 & -2.9698e-15 & 9.7504e-16 & -1.0854e-15 \\ -3.3882e-15 & -8.0989e-16 & -1.2888e-16 & 3.9918e-15 & -2.2404e-16 & -3.9225e-15 & 3.6538e-15 & 3.0759e-15 \\ -2.8912e-15 & -1.6549e-15 & -1.4329e-16 & 4.1303e-16 & 4.5089e-16 & 2.9855e-16 & 2.6684e-15 & 1.7165e-15 \\ 1.4019e-16 & 3.9624e-16 & 8.9394e-16 & 1.6538e-15 & 6.7326e-16 & -7.8393e-16 & -3.6653e-16 & -7.6439e-17 \\ -2.7261e-15 & -7.685e-16 & -1.4098e-15 & -1.8159e-15 & -5.6968e-16 & 5.8426e-16 & 3.3023e-15 & 3.1345e-15 \\ 3.3564e-15 & 8.9708e-16 & -1.7052e-16 & -5.2647e-15 & 3.5883e-16 & 5.9655e-15 & -3.7153e-15 & -2.1352e-15 \\ 1.5856e-15 & -1.4322e-15 & -1.6259e-15 & -3.4782e-15 & -1.2513e-15 & 2.9897e-15 & -1.5507e-15 & -1.861e-15 \end{pmatrix},$$

$$A_0^{\dagger}A_2=\begin{pmatrix} 5.4123e-16 & -6.5226e-16 & 1.36e-15 & 5.1348e-16 & -9.5757e-16 & 8.3267e-17 & 1.0796e-15 & 1.8776e-15 \\ 4.8572e-16 & 2.2204e-16 & 3.6082e-16 & 2.2898e-16 & 1.6653e-16 & -2.0817e-16 & -4.785e-16 & -3.0316e-16 \\ -1.2074e-15 & -6.245e-16 & -1.6098e-15 & 1.6653e-16 & -9.8532e-16 & -6.9389e-16 & 1.4936e-15 & -3.1081e-16 \\ -2.2204e-16 & 2.7756e-16 & 1.249e-16 & 7.7022e-16 & 7.3552e-16 & 9.7145e-16 & -5.094e-16 & -1.0422e-17 \\ -9.0206e-16 & -9.7145e-17 & -2.1927e-15 & -9.2981e-16 & 1.5266e-16 & -3.6082e-16 & -5.8577e-16 & -1.009e-15 \\ 5.8287e-16 & -3.747e-16 & 8.6042e-16 & 6.245e-17 & -4.0246e-16 & -4.0246e-16 & -5.2876e-16 & -4.3644e-16 \\ 6.8001e-16 & 1.4849e-15 & 6.1062e-16 & -1.0547e-15 & 1.4572e-15 & 1.9429e-16 & -6.1587e-16 & -8.8763e-17 \\ 5.4123e-16 & -7.0777e-16 & 3.747e-16 & 3.9552e-16 & -4.4409e-16 & 1.6653e-16 & -8.7313e-16 & 5.3489e-16 \end{pmatrix},$$

$$B_0B_0^{\dagger}=\begin{pmatrix} 1 & 5.5511e-16 & 1.3922e-16 & -1.4988e-15 & -4.7001e-16 & 1.5765e-15 & -1.0547e-15 & -9.992e-16 \\ 4.8392e-16 & 1 & -5.6769e-16 & -7.7716e-16 & -9.948e-16 & 1.5089e-15 & 8.5695e-16 & 7.6328e-16 \\ 8.5258e-17 & -3.8858e-16 & 1 & -1.4433e-15 & 0 & 2.2204e-15 & -4.1633e-17 & 1.6653e-16 \\ 4.7132e-16 & 3.8858e-16 & 2.4029e-16 & 1 & -2.8563e-16 & 1.0003e-15 & -6.4879e-16 & -2.0817e-16 \\ -4.4556e-16 & -3.3307e-16 & -2.8039e-16 & -2.2204e-16 & 1 & -1.4439e-16 & 2.7756e-16 & -1.1102e-16 \\ 9.4234e-16 & 4.5846e-16 & 7.8744e-16 & -4.2166e-16 & -2.3413e-16 & 1 & -9.194e-16 & -3.3307e-16 \\ 3.073e-16 & 4.996e-16 & 0 & 2.1094e-15 & -2.2204e-16 & -2.2204e-15 & 1 & -5.5511e-17 \\ 1.0272e-15 & 4.9803e-16 & -1.3749e-17 & 1.5751e-15 & -8.2236e-16 & -5.5511e-16 & -1.5613e-16 & 1 \end{pmatrix},$$

$$M^{\dagger}NB_0^{\dagger}=\begin{pmatrix} 6.2377e+18 & 2.4105e+18 & 6.1833e+17 & -6.8994e+18 & -4.4277e+18 & 3.3644e+18 & -3.5926e+18 & -6.8943e+18 \\ -1.1248e+19 & -3.834e+18 & -6.6182e+17 & 1.2568e+19 & 8.2455e+18 & -5.573e+18 & 6.1128e+18 & 1.2602e+19 \\ -1.7805e+18 & -7.2581e+17 & -2.1352e+17 & 1.9662e+18 & 1.2463e+18 & -9.9693e+17 & 1.0527e+18 & 1.9558e+18 \\ -4.0065e+16 & -5.6617e+16 & -3.312e+16 & 3.0466e+16 & 1.313e+16 & -4.8279e+16 & 6.2713e+16 & 2.4915e+16 \\ 1.1536e+19 & 3.8397e+18 & 6.0227e+17 & -1.2917e+19 & -8.4948e+18 & 5.6326e+18 & -6.1956e+18 & -1.297e+19 \\ -1.1206e+18 & -1.5149e+17 & 1.3099e+17 & 1.3095e+18 & 9.1978e+17 & -3.3884e+17 & 4.3339e+17 & 1.3485e+18 \\ 5.2048e+18 & 1.8961e+18 & 4.081e+17 & -5.7957e+18 & -3.7609e+18 & 2.6951e+18 & -2.9163e+18 & -5.7955e+18 \\ -1.4725e+18 & -6.206e+17 & -1.748e+17 & 1.6148e+18 & 1.0288e+18 & -8.3326e+17 & 8.839e+17 & 1.6076e+18 \end{pmatrix},$$

$$A_0^{\dagger}FE^{\dagger}=\begin{pmatrix} -2.8568e+16 & 1.6704e+16 & 2.0793e+16 & 5.8446e+16 & 6.7243e+15 & -1.0034e+15 & 4.3475e+16 & -2.8446e+16 \\ 7.7278e+16 & -5.7089e+16 & -1.2901e+17 & -1.5362e+17 & 2.7298e+17 & 9.2832e+16 & -1.8373e+17 & 1.5771e+17 \\ -1.2054e+16 & -5.3499e+15 & -1.9751e+15 & 2.9874e+16 & -1.8516e+16 & -1.0741e+16 & -2.3958e+15 & -6.1126e+15 \\ 7.1856e+15 & 2.3841e+16 & -4.6849e+16 & -4.6068e+16 & 9.1217e+16 & 6.927e+16 & -2.565e+16 & 2.1639e+16 \\ -2.0472e+16 & 2.7534e+15 & 1.6588e+16 & 4.1079e+16 & -5.3058e+16 & -2.28e+16 & 2.1991e+16 & -2.565e+16 \\ 6.2067e+16 & -2.3586e+16 & -1.1923e+17 & -1.498e+17 & 2.1765e+17 & 1.0301e+17 & -1.4548e+17 & 1.2029e+17 \\ 2.023e+16 & -8.1942e+15 & -1.75e+16 & -3.678e+16 & -7.1696e+15 & -1.2602e+15 & -2.6006e+16 & 1.3023e+16 \\ -4.8389e+16 & 6.6122e+16 & 6.6368e+16 & 7.1589e+16 & -1.3094e+17 & -1.4671e+16 & 1.308e+17 & -1.0964e+17 \end{pmatrix},$$

$$L_M=\begin{pmatrix} 1.2979e-13 & -1.4662e-13 & -1.3478e-13 & -4.6119e-14 & -3.7328e-14 & 3.1961e-14 & 0 & 5.6843e-14 \\ -1.0965e-13 & 2.3637e-13 & 3.441e-13 & 6.9303e-14 & 4.244e-14 & 7.2383e-14 & -2.2737e-13 & 0 \\ -5.417e-14 & 5.3952e-14 & 1.5321e-14 & 1.1605e-14 & 1.3894e-14 & -2.3729e-14 & 0 & -5.6843e-14 \\ 7.303e-15 & -4.1083e-15 & 1.8898e-15 & 5.5511e-16 & -2.7367e-15 & 6.1332e-15 & 1.7764e-15 & 1.199e-14 \\ 1.058e-13 & -1.6536e-13 & -3.254e-13 & -4.8524e-14 & -2.2204e-14 & -1.3086e-14 & 1.1369e-13 & -1.1369e-13 \\ -2.5236e-15 & -1.1949e-14 & 3.9934e-14 & 9.594e-15 & -1.3776e-15 & 2.2649e-14 & 7.1054e-15 & 7.816e-14 \\ 1.0633e-13 & -8.1478e-14 & -1.6474e-13 & -5.5747e-14 & 3.8423e-14 & -9.437e-16 & 0 & 0 \\ -3.5282e-14 & 4.1752e-14 & 1.6804e-14 & -1.2359e-14 & 5.7038e-15 & -7.664e-15 & -2.8422e-14 & -7.1054e-14 \end{pmatrix},$$

$$R_E = \begin{pmatrix} 5.6399e-14 & -2.565e-14 & -1.6338e-13 & -1.4923e-13 & 1.4404e-13 & 7.9398e-14 & -1.954e-13 & 1.8119e-13 \\ 5.6917e-14 & -5.9952e-14 & -5.7493e-14 & -6.2536e-14 & 1.2586e-13 & 2.1922e-14 & -9.9476e-14 & 1.2079e-13 \\ 1.5585e-14 & -1.8697e-14 & -1.2879e-14 & 4.7515e-14 & 4.655e-14 & 3.6591e-14 & -1.35e-13 & 2.2027e-13 \\ -1.43e-15 & -5.6567e-15 & -1.2536e-14 & -3.3307e-15 & -3.8836e-14 & 3.4182e-16 & -5.6843e-14 & 1.4921e-13 \\ 1.5677e-14 & -1.9792e-14 & -2.6416e-14 & -8.3162e-15 & 3.7192e-14 & 6.3602e-15 & -2.1316e-14 & 2.1316e-14 \\ 5.8261e-14 & -2.0009e-14 & -3.9568e-14 & -4.9441e-14 & 1.9104e-13 & 8.1823e-14 & -1.8474e-13 & 1.7053e-13 \\ -9.2378e-15 & -4.541e-15 & 2.8567e-14 & -2.0738e-14 & 1.4258e-14 & 4.5358e-15 & 6.7502e-14 & -1.35e-13 \\ -2.7341e-14 & 2.5121e-14 & 3.3505e-14 & 3.3193e-14 & -8.2425e-14 & -1.2146e-14 & 7.1054e-14 & -7.1054e-14 \end{pmatrix}.$$

and $Q_i(i = \overline{1,4})$ and $W_i(i = \overline{1,2})$ are arbitrary matrices over $\mathbb{R}$ with appropriate dimensions, $P_2, Q_2, R_2$ are the forms of (5) when $m = 2$.

## 5. Conclusions

In this paper, we provide the solvability conditions for dual split quaternion matrix Equation (4) and derive the general solution expressions when the equation is consistent. As an application, we give the necessary and sufficient conditions for the existence of a Hermitian solution to Equation (4). Additionally, we analyze some particular instances of dual split quaternion matrix Equation (4). To further demonstrate our findings, an illustrative example is provided. Looking ahead, our research will focus on exploring more intricate matrix and tensor equations over the dual split quaternion algebra.

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