Mathematical Derivation of Existence and Stability of Equation 2-4 Partial Equilibrium Points

Part I. Stability analysis for equations 2

(i) Stability of boundary equilibrium $E_4(P_{1I}^*, I_I{}^*, 0)$

The Jacobian matrix for equation 2 is

$$J = \begin{pmatrix} c_1(1-P_1-I)-\beta I-c_1P_1-m_1 & -(c_1+\beta)P_1 & 0 \\ \beta I & \beta P_1-m_1-d & 0 \\ -(c_1+c_2)P_2 & -c_2P_2 & c_2(1-P_1-P_2-I)-c_1P_1-c_2P_2-m_2 \end{pmatrix}$$

the Jacobi matrix of the equations at $E_4(P_{1I}^*, I_I{}^*, 0)$ is

$$J(E_4) = \begin{pmatrix} c_1\sigma_1 - m_1 + \sigma_2 - \sigma_3 & -d_1 - m_1 - \sigma_3 & 0 \\ -\sigma_2 & 0 & 0 \\ 0 & 0 & c_2\sigma_1 - m_2 - \sigma_3 \end{pmatrix}$$

where $\sigma_1 = \frac{c_1(m_1+d)-\beta(c_1-m_1)}{\beta(\beta+c_1)} - \frac{m_1+d}{\beta} + 1 = \frac{\beta-d}{\beta+c_1}$, $\sigma_2 = \frac{c_1(m_1+d)-\beta(c_1-m_1)}{\beta+c_1}$, $\sigma_3 = \frac{c_1(m_1+d)}{\beta}$. It is easy to determine $\sigma_2 < 0, \sigma_3 > 0$, , and by $J(E_4)$, it has two eigenvalues λ_1, λ_2 satisfying the equation

$$\begin{vmatrix} \lambda - (c_1\sigma_1 - m_1 + \sigma_2 - \sigma_3) & d_1 + m_1 + \sigma_3 \\ \sigma_2 & \lambda \end{vmatrix}$$

i.e., $\lambda^2 - (c_1\sigma_1 - m_1 + \sigma_2 - \sigma_3)\lambda - (d_1 + m_1 + \sigma_3)\sigma_2 = 0$, Combined with the existence condition forE_4,it is easy to see that$\lambda_1 + \lambda_2 = c_1\sigma_1 - m_1 + \sigma_2 - \sigma_3 = -\frac{c_1\beta d + c_1 m_1\beta + c_1^2 m_1 + c_1^2 d}{\beta(\beta+c_1)} < 0$, so $\lambda_1 < 0, \lambda_2 < 0$, both eigenvalues have negative real parts; the third eigenvalue of $J(E_4)$ is $\lambda^* = c_2\sigma_1 - m_2 - \sigma_3 = \frac{(c_2-m_2)\beta^2-(c_1+c_2)\beta d-(m_1+m_2)\beta c_1-(m_1+d)c_1^2}{\beta(\beta+c_1)}$. It is calculated that when

$$c_2 < \frac{[\beta m_2 + c_1(m_1+d)](c_1+\beta)}{\beta(\beta-d)} \triangleq c_{2I}$$

have $\lambda^* < 0$, the equilibrium $E_4(P_{1I}^*, I_I{}^*, 0)$ is locally stable; opposite, E_4 is unstable when $c_2 > c_{2I}$, $\lambda^* > 0$.

(ii) Stability of the internal equilibrium $E_5(P_{1I}^*, I_I^*, P_{2I}^*)$

The eigenvalue of the Jacobi matrix of equations 2 at the positive equilibrium $E_5(P_{1I}^*, I_I^*, P_{2I}^*)$ have three, two of them are λ_1, λ_2, satisfy the characteristic equations

$$\lambda^2 + a_1\lambda + a_2 = 0,$$

where $a_1 = -[c_1(1 - P_{1I}^* - I_I^*) - \beta I_I^* - c_1 P_{1I}^* + \beta P_{1I}^* - 2m_1 - d] = \frac{m_1+d}{\beta} c_1$,

$$a_2 = [c_1(1 - P_{1I}^* - I_I^*) - \beta I_I^* - c_1 P_{1I}^* - m_1](\beta P_{1I}^* - m_1 - d) + (c_1 + \beta) P_{1I}^* \beta I_I^*$$
$$= (c_1 + \beta) P_{1I}^* \beta I_I^*,$$

After calculating, there are$\lambda_1 + \lambda_2 = -a_1 < 0, \lambda_1 \cdot \lambda_2 = a_2 > 0$, so$\lambda_1 < 0, \lambda_2 < 0$. The third eigenvalue is $\lambda_3 = c_2(1 - P_1^* - P_2^* - I^*) - c_1 P_1^* - c_2 P_2^* - m_2$, to make $\lambda_3 < 0$, only $c_2 > c_{2I}$, since the equilibrium $E_5(P_{1I}^*, I_I^*, P_{2I}^*)$ is locally stable, which, combined with its existence, is asymptotically stable as long as E_5 exists. In addition, E_5 as the only internal equilibrium point, local asymptotic stability is also global asymptotic stability.

Part II. Existence and stability of equilibrium of equations 3.

(i) Existence of the endemic disease equilibrium

The endemic disease equilibrium must be the solution of following equation

$$\begin{cases} (c_1 P_1 + \delta c_1 I)(1 - P_1 - I) - \beta P_1 I - m_1 P_1 = 0 \\ \beta P_1 - m_1 - d = 0 \\ c_2 P_2(1 - P_1 - P_2 - I) - (c_1 P_1 + \delta c_1 I) P_2 - m_2 P_2 = 0 \end{cases} \tag{S1}$$

from the second equation of (S1), have $P_{1I}^* = \frac{m_1+d}{\beta}$, substituting P_{1I}^* into the first equation of (S1), have

$$A(I)^2 + BI + C = 0 \qquad \text{(S2)}$$

where $A = \delta c_1 > 0$, $B = \delta c_1(P_{1I}^* - 1) + (c_1 + \beta) P_{1I}^*$, $C = c_1(P_{1I}^*)^2 - (c_1 - m_1) P_{1I}^*$.

Thus equation (S3) has at most two positive roots, for ease of discussion, denoted as

$$f(I) = A(I)^2 + BI + C \qquad \text{(S3)}$$

Clearly,$f(0) = C = \frac{m_1+d}{\beta} \cdot \frac{c_1(m_1+d)-(c_1-m_1)\beta}{\beta}$;$f(1) > 0$, and axis of symmetry $\hat{I}^* = -\frac{B}{2A}$.

It is not difficult to derive the existence of positive roots of equation (S3) as follows,

（1）When $f(0) > 0$ and $\hat{I}^* > 0$, i.e. $\beta < \frac{c_1(m_1+d)}{c_1-m_1}$， $\delta > \frac{(c_1+\beta)(m_1+d)}{c_1(\beta-m_1-d)}$, the number of positive roots of equations (S2) depends on Δ, $\Delta = B^2 - 4AC$.

When $\Delta < 0$,equations (S2) have no positive roots.

When $\Delta = 0$,equations (S2) haveone positive heavy root，

When $\Delta > 0$,equations (S2) have two positive roots.

（2）When$f(0) < 0$,i.e. $\beta > \frac{c_1(m_1+d)}{c_1-m_1}$, (S2) have only one positive root I_+^*;

（3）When $f(0) = 0$ and $\hat{I}^* > 0$, there is $\beta = \frac{c_1(m_1+d)}{c_1-m_1}$， $\delta > \frac{(c_1+\beta)(m_1+d)}{c_1(\beta-m_1-d)}$, equations (S2) have only one positive root;

However, this paper assumes $0 < \delta < 1$, but $\delta > \frac{(c_1+\beta)(m_1+d)}{c_1(\beta-m_1-d)} \geq 1$ when $\beta \leq \frac{c_1(m_1+d)}{c_1-m_1}$, Thus, there is only one case, equation (S2) have only one positive root when $\beta > \frac{c_1(m_1+d)}{c_1-m_1}$, denoted as I_I^*, which substituted into the third equation of equation (S1), have $P_{2I}^* = 1 - \frac{m_2}{c_2} - (1 + \frac{c_1}{c_2})P_{1I}^* - (1 + \delta\frac{c_1}{c_2})I_I^*$. To make it positive, we have

$$c_2 > \frac{c_1(m_1+d)+\beta m_2+\delta c_1\beta I^*}{\beta-m_1-d-\beta I^*} \triangleq c_{2I} \qquad \text{(S4)}$$

Summarizing the above, the existence of the endemic disease equilibrium for the equations 3 as follows:

（1）Equilibrium $E_4(P_1^*, I^*, 0)$ exist when $\beta > \frac{c_1(m_1+d)}{c_1-m_1}$;

（2）Equilibrium $E_5(P_1^*, I^*, P_2^*)$ exist when $\beta > \frac{c_1(m_1+d)}{c_1-m_1}$, $c_2 > c_{2I}$, where$P_{1I}^{*}=\frac{m_1+d}{\beta},I_I^{*}=\frac{-B+\sqrt{\Delta}}{2A}$, $P_{2I}^{*}=1-\frac{m_2}{c_2}-(1+\frac{c_1}{c_2})P_{1I}^{*}-(1+\delta\frac{c_1}{c_2})I_I^{*}$,

$\Delta=B^2-4AC,A,B,C$ see above.

(ii) Stabilization of endemic disease equilibrium

For the equilibrium $E_5(P_1^*,I_I^*,P_2^*)$, which the Jacobi matrix of the equations 3 as

$$J=\begin{pmatrix} c_1(1-P_1^*-I^*)-c_1P_1^*-(\delta c_1+\beta)I^*-m_1 & \delta c_1(1-P_1^*-I^*)-\delta c_1I^*-(c_1+\beta)P_1^* & 0 \\ \beta I^* & \beta P_1^*-m_1-d & 0 \\ -(c_1+c_2)P_2^* & -(\delta c_1+c_2)P_2^* & c_2(1-P_1^*-P_2^*-I^*)-c_1P_1^*-c_2P_2^*-\delta c_1I^*-m_2 \end{pmatrix}$$

the characteristic equation is

$$(\lambda-\lambda^*)f(\lambda)=0 \qquad \text{(S5)}$$

where $\lambda^*=c_2(1-P_{1I}{}^*-P_{2I}{}^*-I_I^*)-c_1P_{1I}{}^*-c_2P_{2I}{}^*-\delta c_1I_I^*-m_2$,

$f(\lambda)=\lambda^2+a_1\lambda+a_2$, $a_1=-c_1(1-P_{1I}{}^*-I_I^*)+c_1P_{1I}{}^*+(\beta+\delta c_1)I_I^*+m_1$,

$a_2=-\beta I_I^*[\delta c_1(1-P_{1I}{}^*-2I_I^*)-(c_1+\beta)P_{1I}{}^*]$.

Clearly, one of the characteristic roots of (S5) is λ^*, and the other two characteristic roots λ_1, λ_1 satisfy

$$\lambda_1+\lambda_2=-a_1, \lambda_1\lambda_2=a_2, \qquad \text{(S6)}$$

which constructs $P_{2\mathrm{I}}{}^*\lambda^*=c_2P_{2\mathrm{I}}{}^*(1-P_{1\mathrm{I}}{}^*-P_{2\mathrm{I}}{}^*-I_I^*)-(c_1P_{1\mathrm{I}}{}^*+\delta c_1I_I^*)P_{2\mathrm{I}}{}^*-m_2P_{2\mathrm{I}}{}^*-c_2(P_{2\mathrm{I}}{}^*)^2=-c_2(P_{2\mathrm{I}}{}^*)^2<0$, so $\lambda^*<0$; from $C=c_1(P_{1I}{}^*)^2-c_1P_{1I}{}^*+m_1P_{1I}{}^*$, we have $a_1=\frac{C}{P_{1I}{}^*}+c_1P_{1I}{}^*+(c_1+\beta+\delta c_1)\ I_I^*$. Combined with the condition for the existence of the equilibrium point$E_5(C<0)$, for $a_1>0$,we need $-\delta(1-P_{1I}^*-I_I^*)-\frac{P_{1I}^*}{I_I^*}=\frac{c_1(1-P_{1I}^*-\ I_I^*)-c_1P_{1I}^*-\beta I_I^*-m_1}{c_1I_I^*}<\delta$, which clearly holds. From $B=\delta c_1(P_{1I}{}^*-1)+(c_1+\beta)P_{1I}{}^*$, so $a_2==-\beta I_I^*(-B-2\delta c_1I_I^*)=\beta I_I^*\sqrt{\Delta}\geq 0$ (I_I^* takes a positive value to ensure its nonnegativity, and the equality sign holds only for $\Delta=0$). Therefore, all eigenvalues have negative real parts, hence equilibrium $E_5(P_1^*,I_I^*,P_2^*)$ is a locally stable.

For the stability of equilibrium $E_4(P_{1I}^*,I_I^*,0)$, equivalent to take $P_{2I}^*=0$ in the above process. Correspondingly, only the first eigenvalue changes to:

$$\lambda_0^* = c_2(1 - P_{1I}{}^* - I_I^*) - c_1 P_{1I}{}^* - \delta c_1 I_I^* - m_2 \qquad \text{(S7)}$$

We have $\lambda_0^* < 0$ when $c_2 < c_{2I}$, and the remaining two eigenvalues λ_1, λ_2 remain unchanged (same as (S6), (S7)), both are less than zero. Hence, $E_4(P_{1I}^*, I_I^*, 0)$.

In summary,

(1) When $c_2 < c_{2I}$, the equilibrium $E_4(P_{1I}^*, I_I^*, 0)$ is locally stable;

(2) The equilibrium $E_5(P_{1I}^*, I_I^*, P_{2I}^*)$ is locally stable as long as it exists.

Part III. Stability of equilibrium points of the Equations 4.

The internal equilibrium $E_5\left(\frac{B_1}{B_0}, \frac{B_2}{B_0}, \frac{B_3}{B_0}\right) = (P_{1I}^*, I_I{}^*, P_{2I}^*)$

where, $B_0 = (c_1 + \beta)(c_1 c + c_2 \beta)$, $B_1 = (c_1 + \beta)(c_2 d + c_2 m_1 - c m_2)$,

$B_2 = c_1 c(c_1 - m_1 + m_2) + (c_1 c_2 - c_2 m_1)\beta - c_1 c_2(m_1 + d)$, $B_3 = (c_2 - m_2)\beta^2 - \mu_1 \beta + \mu_2$, $\mu_1 = c_1 d + c_2 d + c_1 m_1 + c_1 m_2 - c m_2 - c c_1$, $\mu_2 = c c_1 m_1 - c_1^2 m_1 - c_1^2 d$.

Only the stability analysis of E_4, E_5 are given here, the rest of the other equilibrium will not be repeated.

The Jacobian matrix for the Equations 4 is

$$J = \begin{pmatrix} c_1(1-P_1-I) - \beta I - c_1 P_1 - m_1 & -c_1 P_1 - \beta P_1 & 0 \\ \beta I - cI & c(1-P_1-P_2-I) - cI + \beta P_1 - m_1 - d & -cI \\ -(c_1+c_2)P_2 & -c_2 P_2 & c_2(1-P_1-P_2-I) - c_1 P_1 - c_2 P_2 - m_2 \end{pmatrix}$$

as a result, the Jacobi matrix at equilibrium point $E_4 = (P_{1I}^*, I_I{}^*, 0) = (\frac{A_1}{A_0}, \frac{A_2}{A_0}, 0)$ as

$$J(E_4) = \begin{pmatrix} c_1\sigma_1 - m_1 + \frac{\sigma_4}{\sigma_2}\beta - \frac{\sigma_3}{\sigma_2}c_1 & -(\beta + c_1)\frac{\sigma_3}{\sigma_2} & 0 \\ (c - \beta)\frac{\sigma_4}{\sigma_2} & c_1\sigma_1 - m_1 - d + \frac{\sigma_4}{\sigma_2}c + \frac{\sigma_3}{\sigma_2}\beta & \frac{\sigma_4}{\sigma_2}c \\ 0 & 0 & c_2\sigma_1 - m_2 - \frac{\sigma_3}{\sigma_2}c_1 \end{pmatrix}$$

where: $\sigma_1 = \frac{\sigma_4}{\sigma_2} - \frac{\sigma_3}{\sigma_2} + 1 = \frac{\beta(\beta-d)}{\sigma_2}$, $\sigma_2 = \beta c_1 - \beta c + \beta^2 > 0$,

$\sigma_3 = \beta d - \beta c + c_1 d + c_1 m_1 + \beta m_1 - c m_1 > 0$, $\sigma_4 = c_1 d + c_1 m_1 - c m_1 + \beta m_1 - \beta c_1 < 0$. By $J(E_4)$, there is an eigenvalue

$$\lambda^* = c_2\sigma_1 - m_2 - \frac{\sigma_3}{\sigma_2}c_1 = \frac{(c_2-m_2)\beta^2 - \mu_1\beta + \mu_2}{\beta(\beta-c+c_1)},$$

where $\mu_1 = c_1 d + c_2 d + c_1 m_1 + c_1 m_2 - c m_2 - c c_1$, $\mu_2 = c c_1 m_1 - c_1^2 m_1 - c_1^2 d$, if

$$c_2 > \frac{c_1^2(m_1+d) + \beta[c_1(m_1+d) + \beta m_2 - c c_1 - c m_2 + c_1 m_2] - c c_1 m_1}{\beta(\beta-d)} \triangleq c_{2I}$$

then $\lambda^* > 0$. So that the equilibrium $E_4(P_{1I}^*, I_I{}^*, 0)$ is unstable.

The local stability of the equilibrium $E_5 = (\frac{B_1}{B_0}, \frac{B_2}{B_0}, \frac{B_3}{B_0}) = (P_{1I}^*, I_I{}^*, P_{2I}^*)$ is discussed below, and the Jacobi matrix of the equations 4 at $E_5(P_{1I}^*, I_I{}^*, P_{2I}^*)$ is

$$J(P_{1I}^*, I_I{}^*, P_{2I}^*) = \begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix},$$

where $A_{11} = c_1(1 - P_{1I}^* - I_I{}^*) - c_1 P_{1I}^* - \beta I_I{}^* - m_1$, $A_{12} = -c_1 P_{1I}^* - \beta P_{1I}^*$, $A_{21} = -c I_I{}^* + \beta I_I{}^*$, $A_{22} = c(1 - P_{1I}^* - P_{2I}^* - I_I{}^*) - c I_I{}^* + \beta P_{1I}^* - m_1 - d$, $A_{23} = -c I_I{}^*$, $A_{31} = -c_1 P_{2I}^* - c_2 P_{2I}^*$, $A_{32} = -c_2 P_{2I}^*$, $A_{33} = c_2(1 - P_{1I}^* - P_{2I}^* - I_I{}^*) - c_1 P_{1I}^* - c_2 P_{2I}^* - m_2$. Its characteristic equation is:

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$$

Where $a_1 = -(A_{11} + A_{22} + A_{33})$, $a_2 = A_{11}A_{22} + A_{11}A_{33} + A_{22}A_{33} - A_{23}A_{32} - A_{12}A_{21}$, $a_3 = -A_{11}A_{22}A_{33} - A_{12}A_{21}A_{33} - A_{12}A_{23}A_{31} + A_{11}A_{23}A_{32}$,

According to the Routh-Huriwitz criterion, $E_5(P_{1I}^*, I_I{}^*, P_{2I}^*)$ is locally asymptotically stabilized when $a_1 > 0$, $a_3 > 0$, $a_1 a_2 - a_3 > 0$.